Mathematical Reviews

Vol. 18, No. 9 OCTOBER, 1957 pp. 711-784

FOUNDATIONS, THEORY OF SETS, LOGIC

Erdös, P.; and Fodor, G. Some remarks on set theory. V. Acta Sci. Math. Szeged **17** (1956), 250–260.

Let $|E|=\mathfrak{m}$ and, for every $x \in E$, let $f(x)$ be a nonempty subset of E. If $x \in E$, $y \in E$, $x \neq y$, $x \notin f(y)$, and $y \notin f(x)$, then x and y are said to be independent. A subset F of E is called free if F has only one element or else every pair of distinct elements of F are independent. A subset Γ of E is said to have property $T(\mathfrak{q}, \mathfrak{p})$, where $\mathfrak{q} \leq \mathfrak{m}$ and $\mathfrak{p} \leq \mathfrak{m}$, if $|\Gamma|=|\bigcup_{x \in \Gamma} f(x)|=\mathfrak{q}$ and $|\bigcup_{x,y \in \Gamma; x \neq y} [f(x) \cap f(y)]| < \mathfrak{p}$. A subset C of E is called closed; if $f(x) \subseteq C$ for every $x \in C$. Two sets E_1 and E_2 are said to be almost disjoint, if $|E_1 \cap E_2| < \min(|E_1|, |E_2|)$. The authors assume that $|\bigcup_{x \in E} f(x)|=\mathfrak{m}$, and consider the following conditions: (A) There is an $\mathfrak{n}<\mathfrak{m}$ such that $|f(x)|<\mathfrak{n}$ for every $x \in E$. (B) There is an $\mathfrak{n}<\mathfrak{m}$ such that, if $x \in E$, $y \in E$ and $x \neq y$, then $|f(x) \cap f(y)|<\mathfrak{n}$. (C) If $x \in E$, $y \in E$, and $x \neq y$, then $f(x) \not\subseteq f(y)$ and $f(y) \not\subseteq f(x)$. (D) For every $x \in E$, the power of the set of elements $y \in E$ satisfying $f(x) \cap f(y) \neq 0$ is less than $\mathfrak{m}$. Conditions (some involving the generalized continuum hypothesis) are given under which (B), (C), or (D) [for (A), cf. Fodor, Acta Sci. Math. Szeged **16** (1955), 232–240; MR **17**, 951; Erdös, Proc. Amer. Math. Soc. **1** (1950), 127–141; MR **12**, 14] does or does not imply the existence of a subset of E having property $T(\mathfrak{q}, \mathfrak{p})$ for certain values of $\mathfrak{p}$ and $\mathfrak{q}$, or the existence of a free subset of E of power $\mathfrak{f}$ for certain values of $\mathfrak{f}$. For example: If, for every $x \in E$, $|f(x)|<\mathfrak{m}=\aleph_1$, and $\mathfrak{n}<\aleph_0$, then (B) implies the existence of a subset of E having property $T(\mathfrak{m}, \mathfrak{m})$, and the existence of a free subset F of E with $|F|=\mathfrak{m}$, if $\mathfrak{m} \geq \aleph_0$ and $|f(x)|<\mathfrak{m}$ for every $x \in E$, then (B) implies that there exists a free subset F of E with $|F|=\aleph_0$. If $\mathfrak{m}$ is singular and $|f(x)|<\mathfrak{m}$ for every $x \in E$, then (C) does not imply the existence of an independent pair, but, if $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ for every ordinal number α, (B) implies that there exists a free subset F of E with $|F|=\mathfrak{m}$. If $\mathfrak{m}^*$ is the smallest cardinal number such that $\mathfrak{m}$ is the sum of $\mathfrak{m}^*$ cardinal numbers each of which is less than $\mathfrak{m}$, then (D) implies the existence of a free subset F of E with $|F|=\mathfrak{m}^*$. The authors finally deal with the problems of the existence, under condition (A), of a closed proper subset C of E with $|C|=\mathfrak{m}$, and the existence, under (A), of two almost disjoint closed subsets E_1 and E_2 of E with $|E_1|=|E_2|=\mathfrak{m}$. *F. Bagemihl* (Notre Dame, Ind.).

Mostowski, A. On models of axiomatic set-theory. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 663–667.

Let $\Phi(\varepsilon)$ be the conjunction of the axioms of a set theory formalized within the functional calculus of the first order with identity and with ε as the only primitive predicate. $\Phi(R)$ is the statement obtained by replacing in $\Phi(\varepsilon)$ each subformula of the form $u \varepsilon v$ by $\langle u, v\rangle \in R$, where $\langle u, v\rangle$ is the ordered pair of u and v. A relation R is a model for the set theory if and only if $\Phi(R)$ is provable. This paper lists a large number of results about models of set theory, the proofs of which have been or are to be published elsewhere.

P. C. Gilmore.

Cuesta, N. Denjoy orderers. Rev. Mat. Hisp.-Amer. (4) **16** (1956), 179–192. (Spanish)

Let the elements of an ordered set S be the natural numbers. The integers $1, 2, \cdots, n$ appear in a certain order in S; if n is first (in this order), let $a_n = 0$, otherwise let a_n be the immediate predecessor of n. Then the sequence of nonnegative integers $a_1 = 0, a_2, \cdots, a_n, \cdots$, where $a_n < n$ for every n, is uniquely determined by, and uniquely determines, S, and is called the orderer of S. The author's purpose is to study and characterize orderings of the set of natural numbers by means of their orderers. *F. Bagemihl* (Notre Dame, Ind.).

Sierpiński, Wacław. Sur quelques problèmes arithmétiques de la théorie des nombres ordinaux. Czechoslovak Math. J. **6(81)** (1956), 161–163. (Russian summary)

This is a talk, given at a congress of Czechoslovakian mathematicians in 1955, on some statements from elementary number theory which are true for transfinite ordinal numbers. Most of the results given involve either prime ordinal numbers or sums of squares of ordinals, and are already known in the literature. Two unsolved problems are given. These are (1) if α and β are order types and if $\alpha^2\beta^2 = \beta^2\alpha^2$, must $\alpha\beta=\beta\alpha$?; and (2) find all solutions of ordinals of the first kind which satisfy $\xi^2=\eta^3$.

S. Ginsburg (Hawthorne, Calif.).

Finch, Henry Albert. Validity rules for proportionally quantified syllogisms. Philos. Sci. **24** (1957), 1–18.

Maehara, Shôji. Equality axiom on Hilbert's ε-symbol. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1957), 419–435.

By methods like those of Gentzen [Math. Z. **39** (1934), 176–210, 405–431], the author proves that if an axiom system is consistent in the ordinary predicate calculus with equality, then it is also consistent in the predicate calculus with equality extended by adjoining the axiom schemata

$$A(t) \rightarrow A(\varepsilon x A(x)), \tag{1}$$

$$(\forall z)[A(z) \equiv B(z)] \rightarrow \varepsilon z A(z) = \varepsilon z B(z) \tag{2}$$

for the Hilbert ε-symbol or choice operator. For adjunction of (1) only, this result had already been proved in Hilbert and Bernays [Grundlagen der Mathematik, 2 vols., Springer, Berlin, 1934, 1939]. *V. E. Beneš.*

Rasiowa, H. On the ε-theorems. Fund. Math. **43** (1956), 156–165.

This is one of a series of papers in which the author (sometimes jointly with R. Sikorski and A. Mostowski) has expounded the principal results of the lower predicate calculus. It is typical of the algebraic trend in logic of which Miss Rasiowa is a leading representative.

The classical ε-theorems assert (i) that the notion of deducibility in an applied calculus of open formulae is not strengthened by the introduction of quantification,

and (ii) that the adjunction of the ε-functions to a given applied calculus does not strengthen the notion of deducibility in that calculus. The proof of these theorems is achieved here by showing that the given applied calculi are already semantically complete, and hence, that they cannot be strengthened by the inclusion of additional rules or symbols. It is pointed out in the paper that the argument applies also to certain non-classical calculi. When comparing the elegant methods of the present paper with the more laborious proofs of Hilbert and Bernays [Grundlagen der Mathematik, vol. 21, Springer, Berlin, 1939] one should bear in mind that the methods used there are purely syntactical.

A. Robinson (Toronto, Ont.).

Rice, H. G. Recursive and recursively enumerable orders. Trans. Amer. Math. Soc. **83** (1956), 277–300.

On its natural interpretation a recursively enumerable (r.e.) partial order of the natural numbers [cf. Markwald, Math. Ann. **127** (1954), 135–149; MR **15**, 771] is a relation $(Ex)V(n, m, x)$, V recursive, which satisfies the axioms for a partial order, and if this relation is a total order on some subset of the natural numbers it is a r.e. order there. A r.e. relation which is a total order on some recursive subset of the natural numbers is necessarily recursive there. The author restricts himself to (i) recursive orderings of the natural numbers of order type ω, (ii) to a subclass of (i) where there is a recursive bound for the nth element of the order. In this case the order relation $n \prec m$ can be expressed by $f(n) < f(m)$ ('$<$' denotes the natural order) with a recursive one-one f. The author studies the equivalence classes of f corresponding to the same ordering, and order-equivalents, and presents a miscellany of results.

G. Kreisel.

Greniewski, Marek. Utilisation des logiques trivalentes dans la théorie des mécanismes automatiques. I. Réalisation des fonctions fondamentales par des circuits. Com. Acad. R. P. Romîne **6** (1956), 225–229. (Romanian. Russian and French summaries)

Moisil, Gr. C. Ultilisation des logiques trivalentes dans la thoérie des mécanismes automatiques. II. Equation caractéristique d'un relai polarisé. Com. Acad. R. P. Romîne **6** (1956), 231–234. (Romanian. Russian and French summaries)

See also: Matsushima, p. 713; Bagemihl, p. 783.

ALGEBRA

Combinatorial Analysis

Motzkin, T. S.; and Straus, E. G. Some combinatorial extremum problems. Proc. Amer. Math. Soc. **7** (1956), 1014–1021.

Let γ be a given connected graph and $S=\{a_1, a_2, \cdots\}$ a given set of real numbers, where $a_1 \geq a_2 \geq a_3 \geq \cdots$. Let x denote a 1-1 mapping of S onto the set of vertices of γ. The authors consider the problem of finding those mappings x for which a given function $F(x)$ of x is a maximum or minimum. They solve the problem for a case in which no vertex of γ has degree greater than 2 and in which a simple but very restrictive condition is imposed upon $F(x)$. Applications are given to shortest route problems and the theory of continued fractions.

W. T. Tutte (Toronto, Ont.).

Elementary Algebra

Stöhr, Alfred. Neuer Beweis einer Formel über das reelle arithmetisch-geometrische Mittel. Jber. Deutsch. Math. Verein. **58** (1956), Abt. 1, 73–79.

Let $M(a_0, b_0)$ denote the arithmetic-geometric mean of two fixed values a_0, b_0 with $a_0 > b_0 > 0$; thus $M(a_0, b_0)$ is the common limit of the sequences $\{a_n\}$, $\{b_n\}$, where $a_{n+1}=\frac{1}{2}(a_n+b_n)$, $b_{n+1}=(a_nb_n)^{\frac{1}{2}}>0$. Let $c_n=(a_n^2-b_n^2)^{\frac{1}{2}}>0$, so that $c_{n+1}=\frac{1}{2}(a_n-b_n)$. An important formula in the theory of the arithmetic-geometric mean is

$$\lim_{n\to\infty} 2^{-n} \log \frac{4a_n}{c_n} = \frac{\pi}{2} \frac{M(a_0, b_0)}{M(a_0, c_0)},$$

the author now obtains an alternative proof of this formula through the consideration of conformal maps involving elliptic functions.

E. F. Beckenbach.

Marques, Henrique Verol. Principles of equivalence for equations. Gaz. Mat., Lisboa **17** (1956), no. 65, 10–14. (Portuguese)

Kao, R. C.; and Zetterberg, L. H. An identity for the sum of multinomial coefficients. Amer. Math. Monthly **64** (1957), 96–100.

A formula is derived:

$$\mathfrak{S}_{r,n} = \sum (-1)^i \cdot \binom{n}{i} \cdot (n-i)^r \quad (0 \leq i \leq n-1),$$

where $\mathfrak{S}_{r,n}$ stands, for $r \geq n$, for the number of terms of $(x_1+\cdots+x_n)^r$ which contain each of $x_1, \cdots, x_n$ with an exponent ≥ 1. An algebraic proof is given, using a double induction (on n and on r) and also a proof based on probability considerations.

A. J. Kempner.

See also: Ioanin, p. 784.

Linear Algebra

Khan, Nisar A. Some norm inequalities for square matrices. Ganita **6** (1955), 9–14 (1956).

The author uses the Frobenius norm of a matrix A, $\|A\|=(\mathrm{tr}\, AA')^{\frac{1}{2}}=(\sum_{i,j=1}^n a_{ij}\bar{a}_{ij})^{\frac{1}{2}}$, and derives a number of inequalities similar to those established by the reviewer [Amer. Math. Monthly **60** (1953), 173–175; MR **14**, 731] and by Mirsky [ibid. **62** (1955), 428–430; MR **17**, 338].

R. Bellman (Santa Monica, Calif.).

Hegenberg, Leonidas H. B. Sequences and series of matrices. Gaz. Mat., Lisboa **17** (1956), no. 65, 1–5. (Portuguese)

Perfect, Hazel. A remark about canonical forms. Edinburgh Math. Notes no. **40** (1956), 15.

The classical and rational canonical forms of a nilpotent matrix are the same. This fact may be exploited in finding the classical canonical form of a given matrix.

D. E. Rutherford (St. Andrews).

Morgenstern, Dietrich. Eine Verschärfung der Ostrowski'schen Determinantenabschätzung. Math. Z. 66 (1956), 143–146.

Let $A=(a_{ij})$ be an $n\times n$ matrix with real elements satisfying the inequalities

$$s_i=a_{ii}-q_i\geq 0 \quad (i=1, 2, \cdots, n),$$

where $q_i=-|a_{ii}|+\sum_{j=1}^{n}|a_{ij}|$. It is shown that

$$\det A\geq s_1s_2s_3\cdots s_n+q_1s_2s_3\cdots s_n+s_1q_2s_3\cdots s_n+\cdots+s_1s_2\cdots s_{n-1}q_n.$$

W. Ledermann (Manchester).

Robinson, D. W. A proof of the composite function theorem for matric functions. Amer. Math. Monthly 64 (1957), 34–35.

Khan, Nisar A. On involutory matrices. Amer. Math. Monthly 63 (1956), 704–709.

It is proved that the congruence $2^{k-1}+1\equiv 0 \pmod{k}$ holds for no integer $k>1$, and that the congruence $2^{p_1}+2^{p_2}\equiv 0 \pmod{p_1p_2}$ holds for primes p_1 and p_2 only when $p_1p_2=4$ or 6. The properties of involutory matrices A (for which $A^2=I$) proved in this paper, follow directly from known results. The author's theorem 4 which can be written

$$(I\pm A)^k=2^{k-1}(I\pm A)$$

contains his theorems 8, 9 and 10. *H. Gupta.*

Kostarčuk, V. N.; and Pugačev, B. P. Exact estimation of decrease of error in one step of the method of quickest descent. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 2 (1956), 25–30. (Russian)

See also: Nisigaki and Takasu, p. 725; Satake, p. 731; Vala, p. 749; Arcidiacono, p. 756; Downing and Householder, p. 765; Wigner, p. 771; Moisil, p. 784.

Polynomials

Carlitz, L. Some theorems on polynomials. Ark. Mat. 3 (1957), 351–353.

Let $F(x)=x^{2m}+a_1x^{2m-1}+\cdots+a_{2m}$ be a polynomial with rational coefficients. Let p be an odd prime that does not occur in the denominator of any a_r. The author proves the following. (1) If $F(x)$ is congruent (mod p) to the square of a polynomial and p is sufficiently large, then $F(x)$ is the square of a polynomial with rational coefficients. (2) If for all a (mod p) there is an integer b depending on a such that $F(a)\equiv b^2 \pmod{p}$, and p is sufficiently large, then $F(x)$ is the square of a polynomial with rational coefficients. Theorems are also proved for polynomials $F(x)$ with coefficients from GF(q, u), u an indeterminate; and extensions of these theorems are indicated. *M. Newman* (Washington, D.C.).

Zmorovič, V. A. On bounds for roots of algebraic polynomials. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 5(71), 179–183. (Russian)

The author proves two theorems, the first of which was found by Fujiwara [Tôhoku Math. J. 10 (1916), 167–171; cf. Marden, The geometry of the zeros···, Math. Surveys, no. 3, Amer. Math. Soc., New York, 1949, p. 107, ex. 4; MR 11, 101]. The second theorem states that if $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$, $a_0a_n\neq 0$, $n\geq 2$, $\sigma_k=|a_k/a_0|$, then at least k roots of $P(z)$ are outside the disk $|z|\leq(1+\sigma_1+\cdots+\sigma_n)^{-1/(n-k)}$.

R. P. Boas, Jr. (Evanston, Ill.).

★**Schafarewitsch, I. R. Über die Auflösung von Gleichungen höheren Grades (Sturmsche Methode). Mit einem Anhang: Das Hornersche Schema, von H. Karl.** Kleine Ergänzungsreihe zu den Hochschulbüchern für Mathematik, XVII. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. 29 pp.

Translation, by Gero Zschuppe, of the 1954 Russian edition [MR 16, 404].

Manara, C. F. La risoluzione dell'equazione di quinto grado mediante funzioni ellittiche. Period. Mat. (4) 34 (1956), 65–84.

Scopo di questo articolo è dare una esposizione il più possibile elementare ed una illustrazione geometrica dei procedimenti per la rizoluzione dell'equazione generale di 5° grado mediante funzioni ellittiche.

Riassunto dell'autore.

See also: Hamblen, p. 770.

Theory of Invariants

See: Babbage, p. 764.

Continued Fractions

See: Motzkin and Straus, p. 712.

Partial Order Structures

Jordan, Pascual. Beiträge zur Theorie der Schrägverbände. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1956**, 27–42.

Constructions and examples of non-commutative (skew) lattices given in earlier papers are investigated further, with special attention to modularity and distributivity. In particular, Jordan and Witt [same Abh. **1953**, 223–232; MR 15, 595] showed a method of constructing a skew-lattice B from a given lattice A; this construction is now applied in case A is also skew, and the persistence in B of axioms of A is investigated. New topics are: idempotent functions on a skew-lattice; skew-lattices S in which with each $a\in S$ is associated an $\bar{a}\in S$ with $\bar{\bar{a}}=a$ and $\overline{a\wedge b}=\bar{b}\vee\bar{a}$; and various forms of the distributive law.

P. M. Whitman (Silver Spring, Md.).

Matsushima, Yataro. On the B-covers in lattices. Proc. Japan Acad. 32 (1956), 549–553.

If a and b are any two elements of a lattice L, then the author defines the B-cover, $B(a, b)$, of a and b as the set-theoretic intersection of the sets

$$J(a, b)=\{x|x=(a\cap x)\cup(b\cap x)\},$$
$$C(J(a, b))=\{x|x=(a\cup x)\cap(b\cup x)\}.$$

If a, b and c are any three elements of L, then acb means $c\in B(a, b)$. The author proves several properties of B-covers and of the relation acb. If L is modular, then $B(a, b)$ is a sublattice of L, but this need not be true if L is not modular. If abc, axb, byc, then (1) xby in any lattice and (2) axc, ayc, xyc, axy in a modular lattice. L is distributive, if and only if

the conditions $x \in B(a, b)$ and $a \cap b \leq x \leq a \cup b$ are equivalent for any two elements a and b. In any lattice the condition $B(a, b) = B(c, d)$ implies $a \cup b = c \cup d$ and $a \cap b = c \cap d$ for any a, b, c and $d \in L$. L is distributive if and only if $a \cup b = c \cup d$ and $a \cap b = c \cap d$ imply $B(a, b) = B(c, d)$. This result is a generalisation of a result obtained by L. M. Kelly [Duke Math. J. **19** (1952), 661–669; MR **14**, 494]. Finally the author proves, that if L is generated by two maximal chains

$$a = a_n > a_{n-1} > \cdots > a_1 = a \cap b,$$
$$b = b_m > b_{m-1} > \cdots > b_1 = a \cap b,$$

then $B(a, b)$ consists of mn lattice points. *P. Dwinger.*

Mitrinovitch, Dragoslav S. Sur une démonstration dans l'algèbre de Dubreil. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. **10** (1956), 3 pp. (Serbo-Croatian summary)

Monteiro, Antonio. Characteristic properties of the filters of a Boolean algebra. Acta Cuyana Ingen. **1** (1954), no. 5, 6 pp. (Spanish)

Eine Teilmenge S eines distributiven vollständigen Verbandes R mit dem Nullelement o und dem Einselement e heißt kompakt, wenn jede Teilmenge P von S mit $\bigcap\{p : p \in P\} = o$ eine endliche Teilmenge Q mit $\bigcap\{p : p \in Q\} = o$ enthält. S wird als eine Basis von R bezeichnet, wenn jedes Element von R ein Produkt von Elementen von S bildet. Mit Hilfe von Darstellungssätzen von M. H. Stone [Trans. Amer. Math. Soc. **41** (1937), 375–481] und H. Wallman [Ann. of Math. (2) **39** (1938), 112–126] zeigt der Verf., daß R dann und nur dann dem Verband aller Filter einer Booleschen Algebra isomorph ist, wenn R eine kompakte Basis besitzt, deren Elemente Komplemente in bezug auf e haben. R ist dann selbst kompakt.

K. Krickeberg (Würzburg).

See also: Suzuki, p. 715; Tamura, p. 717; Mazurkiewicz, p. 768; Moisil, p. 784.

Rings, Fields, Algebras

Popovici, Constantin P. Propriétés locales des entiers de Gauss. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 11–20. (Romanian. Russian and French summaries)

Let $\mathfrak{J}[i]$ be the ring of Gaussian integers and let α, β, π belong to it. α is said to be divisible by $\beta \neq 0$ locally, or with respect to π, if there exists a $z \in \mathfrak{J}[i]$, such that $z\alpha/\beta \in \mathfrak{J}[i]$. Clearly, this simply means that $\pi^j | \beta$ implies $\pi^j | \alpha$. Divisibility with respect to π in $\mathfrak{J}[i]$ is equivalent to ordinary divisibility in the ring $\mathfrak{J}[i]/\pi$, whose elements are $\pi^\rho \alpha'/\beta'$, $\rho \geq 0$, $(\alpha', \pi) = (\beta', \pi) = 1$. From this remark all properties of "local" divisibility (essentially the same as those of ordinary divisibility) immediately follow.

E. Grosswald (Philadelphia, Pa.).

Northcott, D. G. On irreducible ideals in local rings. J. London Math. Soc. **32** (1957), 82–88.

Let Q be a local ring of dimension d ($d \geq 1$) and $\mathfrak{m}$ the maximal ideal of Q. The local ring Q is said to be semi-regular if whenever $v_1, \cdots, v_d$ is a system of parameters, the ideal $(v_1, \cdots, v_{d-1})$ does not have $\mathfrak{m}$ as one of its prime ideals. Suppose Q is a semi-regular local ring and $\mathfrak{q}$ and $\mathfrak{q}'$ are ideals in Q each of which is generated by a system of parameters. It is shown that if $\mathfrak{q} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ and $\mathfrak{q}' = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{n'}$ are irredundant representations of $\mathfrak{q}$ and $\mathfrak{q}'$ as intersections of irreducible $\mathfrak{m}$-primary ideals, then $n = n'$. Since every regular local ring is semi-regular, it is pointed out that the following result of Gröbner [Monatsh. Math. **55** (1951), 138–145; MR **13**, 202] is obtained as a special case: in a regular local ring each ideal generated by a system of parameters is irreducible.

M. Auslander (Princeton, N.J.).

Pursell, Lyle E. An algebraic characterization of fixed ideals in certain function rings. Pacific J. Math. **5** (1955), 963–969.

Let X be a regular Hausdorff space, D a division ring. Consider rings $R \subset D^X$ such that (1) every

$$Z(f) = \{x \in X | f(x) = 0\},$$

$f \in R$, is closed; (2) if $F \subset X$ is closed, $x \in X - F$, then for some $f \in R$, $f(x) \neq 0$, $Z(f)$ is a neighborhood of F; (3) if $F \subset X$ is closed, $f \in R$, $Z(f)$ does not intersect F, then, for some $g \in R$, $fg = 1$ on F; (4) for any $x \in X$, $(x) = Z(f)$ for some $f \in R$. An ideal $J \subset R$ is called fixed if, for some $x \in X$, $f(x) = 0$ for every $f \in I$. A characterization of fixed ideals is given in terms of the algebraic structure of R. From this, the main theorem is easily deduced: if $R \subset D^X$, $R_1 \subset D_1^{X_1}$ satisfy (1)–(4) and are isomorphic, then X, X_1 are homeomorphic. As a corollary, if every point of X, X_1 is G_δ and the rings of all continuous real-valued functions on X and X_1 are isomorphic, then X, X_1 are homeomorphic. By an extension of the above method, it is proved that, under certain broad conditions, two r-differentiable manifolds admit of a differentiable homeomorphism whenever the respective rings of r-fold continuously differentiable functions are isomorphic.

M. Katětov (Prague).

★ **Terada, Fumiyuki. A generalization of the principal ideal theorem.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 264–265. Science Council of Japan, Tokyo, 1956.

This paper has appeared previously in greater detail [J. Math. Soc. Japan **7** (1955), 530–536; MR **18**, 644].

★ **Moriya, Mikao. Zusammenhang zwischen 2-Kohomologiegruppe und Differente.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 247–249. Science Council of Japan, Tokyo, 1956.

This article has appeared previously in greater detail under the title "Zusammenhang zwischen Derivationsmodul und 2-Kohomologiegruppe. I" [J. Math. Soc. Japan **7** (1955), 444–452; MR **18**, 8].

Gel'fand, I. M.; Raĭkov, D. A.; and Šilov, G. E. Commutative normed rings. Amer. Math. Soc. Transl. (2) **5** (1957), 115–220.

A translation of the 1946 Russian paper reviewed in MR **10**, 258.

Herstein, I. N. Lie and Jordan systems in simple rings with involution. Amer. J. Math. **78** (1956), 629–649.

The author's study of the Lie and Jordan structure of simple (associative) rings [same J. **77** (1955), 279–285; Duke Math. J. **22** (1955), 471–476; MR **16**, 789; **17**, 577] is continued for simple rings with involution.

Let A be a simple ring of characteristic $\neq 2$ with center Z. Let S and K be respectively the sets of self-adjoint and skew elements relative to an involution on A. It is shown that S is a simple Jordan ring and, if $Z=0$ or $(A:Z)>4$, then $\bar{S}=\bar{K}=A$ where $\bar{M}$ denotes the subring generated by M. The principal theorem of the paper is that, if $Z=0$ or $(A:Z)>16$, then any Lie ideal U of K satisfies either $U\subseteq Z$ or $U\supseteq[K,K]$. It follows that, if $Z=0$ or $(A:Z)>16$ then $[[K,K],[K,K]]=[K,K]$.

R. D. Schafer (Storrs, Conn.).

★ **Narita, Masao. On the structure of complete local rings.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 251–253. Science Council of Japan, Tokyo, 1956.

Fuller proofs of theorems stated in this article have been published previously [J. Math. Soc. Japan **7** (1955), 435–443; MR **18**, 6] when the same work appeared in greater detail. No new references are given.

Smiley, M. F. Jordan homomorphisms onto prime rings. Trans. Amer. Math. Soc. **84** (1957), 426–429.

The author presents a new brief self-contained proof of the following theorem of Herstein [same Trans. **81** (1956), 331–341; MR **17**, 938]. Let $a\to a'$ be a mapping from a ring R onto a prime ring R' of characteristic not two such that $(ab+ba)'=a'b'+b'a'$ and $(a+b)'=a'+b'$ for all a, $b\in R$. Then either $(ab)'=a'b'$ for all a and b or $(ab)'=b'a'$ for all a and b. Herstein's assumption that the characteristic of R' be different from three turns out to be unnecessary. The proof is based on the following new result on prime rings. Let P be a prime ring such that $(xy-yx)^2=0$ for all x and y. Then P is commutative

C. W. Curtis (Madison, Wis.).

Merriell, D. M. Flexible almost alternative algebras. Proc. Amer. Math. Soc. **8** (1957), 146–150.

The definition of an almost alternative algebra [Albert, Portugal. Math. **8** (1949), 23–36; MR **11**, 316] involves parameters α, β, γ, etc., in terms of which the results of this paper are stated. Let A be an almost left alternative algebra over F of characteristic $\neq 2$. If $\alpha-\beta-\gamma+\delta\neq 0$, then A is Lie-admissible. If A is flexible but not alternative, then $(\alpha-\beta+\gamma-\delta)(\alpha+\beta-1)=\beta$. If A is flexible with $\alpha+\beta\neq 3/4$, then A is quasiequivalent over a scalar extension K of F to an alternative algebra $B=A_K(\lambda)$.

R. D. Schafer (Storrs, Conn.).

Higman, D. G. Relative cohomology. Canad. J. Math. **9** (1957), 19–34.

Let R and S be rings with identity elements (assumed to act as the identity maps on all modules under consideration). Let χ be a homomorphism of S into R mapping the identity element of S onto that of R. Every R-module is regarded simultaneously as an S-module, via χ. Let M and N be R-modules, and let $K(M)$ denote the kernel of the canonical R-epimorphism $R\otimes_S M\to M$. The injection $K(M)\to R\otimes_S M$ yields the homomorphism

$$\mathrm{Hom}_R(R\otimes_S M, N)\to\mathrm{Hom}_R(K(M), N).$$

Since

$$\mathrm{Hom}_R(R\otimes_S M, N)\approx\mathrm{Hom}_S(M, N),$$
$$\mathrm{Hom}_R(K(M), N)\subset\mathrm{Hom}_S(K(M), N),$$

this defines a homomorphism

$$\delta_{M,N}:\mathrm{Hom}_S(M, N)\to\mathrm{Hom}_S(K(M), N).$$

Iterating this construction, forming $\delta_{K(M),N}$ etc., the author obtains a complex whose cohomology groups $H^i(M, N)$ $(i=0, 1, \cdots)$ are the analogues, for an S-relative R-module theory, of the groups $\mathrm{Ext}_R{}^i(M, N)$ of Cartan and Eilenberg, "Homological algebra" [Princeton, 1956; MR **17**, 1040]. In particular,

$$H^0(M, N)=\mathrm{Hom}_R(M, N).$$

In addition, the author gives a dual construction yielding isomorphic groups $\bar{H}^i(N, M)\approx H^i(M, N)$. This construction starts with the R-epimorphism

$$\mathrm{Hom}_S(R, N)\to\mathrm{Hom}_S(R, N)/\mathrm{Hom}_R(R, N)=L(N),$$

yielding the homomorphism

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(R, N))\to\mathrm{Hom}_R(M, L(N)).$$

Since

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(R, N)\approx(\mathrm{Hom}_S(M, N),$$
$$\mathrm{Hom}_R(M, L(N))\subset\mathrm{Hom}_S(M, L(N)),$$

this yields a homomorphism

$$\delta_{N,M}:\mathrm{Hom}_S(M, N)\to\mathrm{Hom}_S(M, L(N)).$$

Iteration of this construction yields a complex whose cohomology groups are the $\bar{H}^i(N, M)$. These two constructions correspond to the calculations of $\mathrm{Ext}_R{}^i(M, N)$ from a projective resolution of M or from an injective resolution of N, respectively.

The author then proceeds to investigate the significance of these groups for a relative module extension theory. Also, he obtains some formal properties of his cohomology groups, notably in special cases involving double modules. He gives some results on the corresponding notion of relative cohomological module dimension in the case where χ: $S\to R$ satisfies the self-duality condition that there is an R-isomorphism between

$$R\otimes_S M \text{ and } \mathrm{Hom}_S(R, M),$$

for all M. The groups $H^i(M, N)$ are modules over the center of R, and the author finally studies their annihilators in the center of R.

The general theory given here is essentially equivalent to that given by the reviewer [Trans. Amer. Math. Soc. **82** (1956), 246–269; MR **18**, 278] but was obtained independently.

G. P. Hochschild.

See also: Carlitz, p. 713; Cohen, p. 718; Rizza, p. 725; Schütte, p. 755; Moisil, p. 784.

Groups, Generalized Groups

★ **Suzuki, Michio. Structure of a group and the structure of its lattice of subgroups.** Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 10. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. 96 pp. DM 16.50.

This monograph deals with the question of the relations between the structure of a group and the structure of its lattice of subgroups, concerning which an extensive amount of information has been obtained, beginning with the results of I. Rottlander, O. Ore and R. Baer. Within the limits imposed by the nature of the Ergebnisse reports, the author gives a connected, and largely self-contained account of the presents state of the subject, taking for granted those parts of the standard theories of groups and lattices to be found, for example, in the

books of Zassenhaus [Lehrbuch der Gruppentheorie, Bd. 1, Teubner, Leipzig-Berlin, 1937] and Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR 10, 673]. He includes several of his hitherto unpublished results, and mentions several unsolved problems, a particularly interesting example which is quoted below.

The first of the four chapters, entitled Groups with a special kind of subgroup lattice, contains, among other things, K. Iwasawa's characterizations of finite modular groups and modular groups with elements of infinite order.

The second chapter, Isomorphsims of subgroup lattices, is concerned with the problem: When is a given projectivity (i.e., lattice isomorphism) of one group onto another induced by a group isomorphism, and more generally, when does the existence of a projectivity imply the existence of a group isomorphism? One of the most interesting of the open questions mentioned by the author occurs here. If a finite, non-abelian simple group G is lattice isomorphic with a group H, then H is a non-abelian simple group of the same order. Are G and H necessarily isomorphic? The author has proved that if there exists a projectivity of $G\times G$ onto a group K then K is isomorphic with $G\times G$.

The third chapter, Homomorphisms of Subgroup Lattices, is concerned with such results as the characterization due to Whitman, Zappa and the author, of groups whose subgroup lattices admit complete lattice homomorphisms onto the subgroup lattice of a cyclic group, and the determination due to Zappa and others of group homomorphisms which induce lattice homomorphisms. It is proved that a finite solvable group admits a lattice homomorphism onto a group H only if H is solvable.

A group H is called a dual of a group G if there exists a dualism of G onto H, i.e. a dual isomorphism of the lattice of subgroups of G onto that of H. The final chapter, Dualisms of subgroup lattices, contains Baer's characterization of abelian groups with duals, and the author's characterization of nilpotent and finite solvable groups with duals.

The reader of this book will be able to judge the extent to which the theory of lattices has contributed to date to the theory of groups, aside from serving as a source of problems. *Donald G. Higman* (Ann Arbor, Mich.).

Wiegold, J. Groups with boundedly finite classes of conjugate elements. Proc. Roy. Soc. London. Ser. A. 238 (1957), 389–401.

A group is said to be n-BFC, where n is a positive integer, if it has a conjugate class containing n elements, but no larger conjugate class. The results proved in this paper are (i) if a group is n-,BFC then its derived group has order at most n^N, where N is an explicitly given function of n of the order of $\frac{1}{4}n^4 (\log_2 n)^3$, and (ii) if n is prime, the derived group of an n-BFC group is cyclic of order p. There is also some discussion of the case $n=4$; here the derived group is at worst elementary abelian of order 8, but the proof is given only in part.

Graham Higman (Oxford).

Petresco, Julian. Sur les groupes libres. Bull. Sci. Math. (2) 80 (1956), 6–32.

We consider finite sequences $r_1, \cdots, r_n$ of letters. An interval is a subsequence of r_k with $i\leq k\leq j$, and r_i and r_j are the extremities of this subsequence. A segmentation of a sequence is a set of subsequences such that every r_i $(i=1, \cdots, n)$ is an extremity of exactly one segment. A segmentation is said to be concordant if any two subsequences are disjoint or one lies inside the other. A concordant segmentation is said to be unitary if the $r_i\neq 1$ are elements of a group G and if every subsequence, regarded as a product in G is the identity of G. It is shown that a unitary sequence is the identity in every group and hence the author has a criterion for identities in free groups.

Finitely generated subgroups of free groups are studied and the Nielson properties of sets of elements are investigated, together with the notion of a center of a word in a free group.

Let $v_1, v_2, \cdots, v_m$ be elements of a free group F, and replace these by $u_k=n_k^-$, $k\neq j$, $u_j=w_1v_jw_2$ where w_1 and w_2 are words in the v_i, $i\neq j$, and their inverses. Call this a simple transformation. It is shown that if the v's are not free generators of the subgroup which they generate, then there is a simple transformation reducing the total length of the elements. *Marshall Hall, Jr.* (Columbus, Ohio).

★**Brauer, Richard. Number theoretical investigations on groups of finite order.** Proceedings of the international symposium on algebraic number theory, Tokyo and Nikko, 1955, pp. 55–62. Science Council of Japan, Tokyo, 1956.

This paper gives the substance of a lecture on the modular representation theory of a group G of finite order g, in which seven conjectures or problems are discussed. We give four of these as being more explicit than the rest. (I) Are the values of the decomposition numbers d_{ij} of a block B bounded by a number which depends only on p and the defect δ? (II) What is the group theoretic significance of the number of p-blocks of defect zero? (III) If the exact power of p dividing the degree z of an irreducible representation of G belonging to B of defect δ is given by

$$e(z)=a-\delta+\varepsilon,$$

where $g=p^a g'$, $(g, g')=1$, is it true that $\varepsilon>0$ if and only if the defect group D of B is non-abelian? (Note the misprint.) (V) If f is the degree of a modular irreducible representation belonging to B, is it true that $e(f)\leq a$?

G. de B. Robinson (Toronto, Ont.).

Neumann, B. H. On a conjecture of Hanna Neumann. Proc. Glasgow Math. Assoc. 3 (1956), 13–17.

An example is given, viz. the Sylow 2-subgroup of the symmetric group of degree 8, of a non-metabelian 3-generator group all of whose 2-generator subgroups are metabelian. An example is also given of a non-metabelian 4-generator group, of order 2^{14}, all of whose 3-generator subgroups are metabelian. Both examples are groups of exponent 8. *P. Hall* (Cambridge, England).

Osima, Masaru. On the representations of the generalized symmetric group. II. Math. J. Okayama Univ. 6 (1956), 81–97.

In a former paper [same J. 4 (1954), 39–56; MR 16, 794] the author gave the ordinary representation theory of the generalized symmetric group $S(n, m)$. Here he studies the modular theory of $S(n, m)$, giving first some general results which are of interest in themselves.

In I the author showed that the irreducible representations $[\alpha]^*$ of $S(n, m)$ are in 1-1 correspondence with the skew Young diagrams $[\alpha_0]\cdot[\alpha_1]\cdots[\alpha_{m-1}]$, where $[\alpha_i]$ is a right diagram with n_i nodes and $\sum n_i=n$. In

Theorem 1 of II he proves that two representations $[\alpha^*]$ and $[\beta]^*$ with $(m, p)=1$ belong to the same-block if and only if $[\alpha_i]$ and $[\beta_i]$ are of the same weight and have the same p-core for $i=0, 1, \cdots, m-1$. The number of ordinary representations in a block B of $S(n, m)$ is calculated in the same way as for S_n (Theorem 2), and the defect group is the direct product of the defect groups of the S_{n_i} (Theorem 3). The 1-1 association of the modular representations of S_n with the regular diagrams [Robinson and Taulbee, Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 596–598; MR 17, 126] is utilized to yield an enumeration of modular representations in B similar to that already given for S_n.

The most interesting result of the paper is contained in Theorem 5 which reads as follows. Let B be a block of $S(n,m)$ with $(m, p)=1$ which contains an irreducible representation $[\alpha]^*$ and B_i the block of S_{n_i} which contains $[\alpha_i]$. The decomposition matrix D of B is given by the Kronecker product $D=D_0\times D_1\times\cdots\times D_{m-1}$ where D_i is the decomposition matrix of B_i.

The case where p/m can be similarly handled but requires more careful analysis. *G. de B. Robinson.*

Reiner, Irving. Integral representations of cyclic groups of prime order. Proc. Amer. Math. Soc. 8 (1957), 142–146.

Let Z denote the ring of rational integers, $Z[g]$ the group ring of the cyclic group $\{g\}$ of prime order p, and θ a primitive pth root of unity. If M is any $Z[g]$-module then the submodule M_s annihilated by $s=1+g+g^2+\cdots+g^{p-1}$ is operator isomorphic to a direct sum $Z[\theta]\oplus\cdots\oplus Z[\theta]\oplus\mathfrak{A}$ where $\mathfrak{A}$ is an ideal of $Z[\theta]$. The author proves that M is completely characterized by the following set of invariants: the ideal class of $\mathfrak{A}$, the rank of M_s as $Z[\theta]$-module, and the ranks of M/M_s and $(g-1)M/(\theta-1)M_s$ as Z-modules. A corollary states that if h is the class number of $Z[\theta]$ then $Z[g]$ has $2h+1$ unimodularly in-equivalent integral representations. *H. K. Farahat* (Sheffield).

Bertaut, E. F. Les groupes de translation non primitifs et la méthode statistique. Acta Cryst. 9 (1956), 322.

It is shown how the statistical method for determining the signs of the crystal structure factors may be extended to those centrosymmetric space groups for which the unit cell is chosen to be non-primitive. *H. A. Hauptman.*

Preston, G. B. The structure of normal inverse semigroups. Proc. Glasgow Math. Assoc. 3 (1956), 1–9.

In a series of previous papers [J. London Math. Soc. 29 (1954), 396–403, 411–419; MR 16, 215, 216] the author has defined a normal inverse subsemigroup N of an inverse semigroup S in such a way that N is normal if and only if it is the kernel of a homomorphism of S. Here, the structure of such normal inverse subsemigroups is investigated. If an inverse semigroup N is the union of disjoint inverse semigroups N_α and if E and E_α are the sets of idempotents in N and N_α respectively, then a necessary and sufficient condition that N can be a normal subsemigroup of an inverse semigroup is that E be a semilattice with elements E_α. This is equivalent to the condition that N be a semilattice of inverse subsemigroups, the elements of the semilattice being N_α. Any semigroup that is a semilattice of inverse semigroups is itself an inverse semigroup. The structure of a normal inverse semigroup N, i.e. of a semilattice of inverse semigroups, is shown to be determined to within isomorphism by the semilattice, the elements of the semilattice, and a certain set of mappings of N. *D. C. Murdoch.*

Iséki, Kiyoshi. A characterisation of regular semi-group. Proc. Japan Acad. 32 (1956), 676–677.

This paper consists of the following interesting and neatly-proved theorem: — A semi-group is regular if and only if $AB=A\cap B$ for every right-ideal A and every left-ideal B. This is similar to a characterization of regular rings by L. Kovács [Publ. Math. Debrecen 4 (1956), 465–468; MR 18, 188]. There are a couple of corollaries.

H. A. Thurston (Bristol).

Tamura, Takayuki. The theory of construction of finite semigroups. I. Osaka Math. J. 8 (1956), 243–261.

A decomposition of a semi-group into congruence-classes is called an s-decomposition if the factor semi-group is a semi-lattice; a c-decomposition, if idempotent. The author actually defines a more general notion, which he calls μ-decomposition and which includes these two as special cases. He proves that any semi-group has a greatest μ-decomposition, where a decomposition is said to be greater than another if it is a refinement thereof.

The first structure theorem, and the last theorem in the present paper, is that a finite s-indecomposable semi-group is either: (i) a c-indecomposable semi-group but not a group, (ii) a unipotent semi-group with a zero, (iii) a unipotent semi-group without a zero, or (iv) a c-decomposable, non-commutative, non-unipotent semi-group.

There are to be five sequels. *H. A. Thurston.*

See also: Tchudakoff, p. 719; Yokota, p. 754.

THEORY OF NUMBERS

General Theory of Numbers

Carlitz, L. A note on Gauss' "Serierum singularium". Portugal. Math. 15 (1956), 9–12.

By using an identity of Euler [see Hardy and Wright, Introduction to the theory of numbers, 2nd ed., Oxford, 1945, Th. 349, p. 278] the author deduces in a natural way two formulas of Gauss. One of these, for example, is

$$\sum_{r=0}^{2m}\begin{bmatrix}2m\\ r\end{bmatrix}(-1)^r=(1-x)(1-x^3)\cdots(1-x^{2m-1}),$$

where

$$\begin{bmatrix}m\\ r\end{bmatrix}=(x)_m/\{(x)_r(x)_{m-r}\},$$
$$(a)_m=(1-a)(1-ax)\cdots(1-ax^{m-1}).$$

The use of a more general identity leads to two generalized formulas. *W. H. Gage* (Vancouver, B.C.).

Wall, C. T. C. A theorem on prime powers. Eureka no. 19 (1957), 10–11.

The only (non-prime) prime powers which differ by unity are 8 and 9. The proof is extremely short.

Kanold, Hans-Joachim. Über einen Satz von L. E. Dickson. II. Math. Ann. 132 (1956), 246–255.

The present paper contains the proof of a theorem a more special form of which had been conjectured by the author in an earlier publication [Math. Ann. **131** (1956), 167–179; MR **17**, 1185]. Let k, N, r, Z be natural numbers, $(N, Z)=1$, and let $\mathfrak{N}$ be the set of positive integers n such that $\sigma_r(n)/n^r=Z/N$ and $V(n)=k$, where $\sigma_r(n)$ denotes the sum of the rth powers of all positive divisors of n and $V(n)$ the number of distinct prime factors of n. The main object of the paper is to establish under what circumstances $\mathfrak{N}$ is an infinite set. The author shows that this is the case if and only if the following four conditions are satisfied. (i) $r=1$; (ii) $Z\equiv 0 \pmod 2$; (iii) there exists an odd number m such that $\sigma_1(m)/m=Z/2N$ and $V(m)=k-2$; (iv) there exist infinitely many even perfect numbers. The truth of (iv) is not, of course, known; and it may be hoped that an investigation such as that under review may help to elucidate this question. *L. Mirsky* (Sheffield).

Buquet, A. Démonstration élémentaire du théorème de Mordell-Weil pour l'équation diophantienne en nombres rationnels $X(X^2+CX+D)=Z^2$. Mathesis 65 (1956), 379–390.

It was established by L. J. Mordell [Proc. Cambridge Philos. Soc. **21** (1922), 179–192] that all the rational points on the cubic curve $4x^3-g_2x-g_3=y^2$ could be found by the chord and tangent process starting from a finite number of them. In this paper the author gives an elementary proof of this theorem for the title equation. It is shown that the basic solutions can be selected among four types of points, the classification of which is too complicated to be given here. Use is made of some results due to Rignaux concerning the diophantine equation $x^4+Cx^2y^2+Dy^4=z^2$. At last examples are given.
W. Ljunggren (Blindern).

Thébault, V. Equations diophantiennes. Mathesis 65 (1956), 421–423.

The author gives special parametric solutions of four diophantine equations. As an example we mention the equation $ab(a+b)=cd(c+d)$ with the solutions

$$a=xy(x+y),\ b=(z-x-y)(xy+xz-z^2),$$
$$c=x(z^2-xy-xz),\ d=y[(x-z)^2-yz].$$

At last some solutions of $A^4+B^4+C^4=D^4+E^4$ are noted. *W. Ljunggren* (Blindern).

★ **Xeroudakes, George. The Diophantine system**
$$\sum A_i=\sum B_i,\ \sum A_i^3=\sum B_i^3 (i=1, 2, 3).$$
Applications and significant conclusions. (Greek). Athens, 1955. 48 pp.

Chalk, J. H. H. An estimate for the fundamental solutions of a generalized Pell equation. Math. Ann. 132 (1956), 263–276.

The author applies a method, used by Schur to obtain an upper bound for the smallest solution of Pell's equation, to the equation

$$x^2+y^2-D(z^2+w^2)=1,$$

where D is any positive integer. He shows that there exist integers x, y, z, w satisfying this equation for which

$$(x^2+y^2)^{\frac{1}{2}}\leqq 1+D\prod_{\substack{p|D\\ p<2}}\left\{1+\left(\frac{-1}{p}\right)\frac{1}{p}\right\}.$$

Use is made of the group of proper automorphs of certain hermitian forms and the corresponding group of bilinear transformations which is Fuchsian of the first kind (horocyclic). Where Schur used Kronecker's form of the class-number formula, the author uses an identity of Humbert involving integrals over fundamental regions whose integrands involve the forms f_1 $(=x\bar{x}-Dy\bar{y})$, $f_2, \cdots, f_h$ which are properly primitive of determinant D, and no two of which are properly equivalent. The inequality (1) is obtained by ignoring the contributions from all the forms other than the first and by replacing the fundamental region by a smaller region outside a certain circle. By a more careful analysis, involving an application of the Siegel-Tsuji theorem on the non-Euclidean area of fundamental regions, the inequality (1) is sharpened by reducing the magnitude of the right-hand side by about 5 per cent. *R. A. Rankin* (Glasgow).

Cohen, Eckford. Simultaneous pairs of linear and quadratic equations in a Galois field. Canad. J. Math. 9 (1957), 74–78.

Let $N_s(m, n)$ denote the number of solutions $x_1, \cdots, x_s$ of the system

$$\alpha_1x_1^2+\cdots+\alpha_sx_s^2=m,\ \beta_1x_1+\cdots+\beta_sx_s=n,$$

where x_i, α_i, β_i, m, $n \in \mathrm{GF}(p^r)$, $p>2$, $\alpha_i\neq 0$, $\beta_i\neq 0$. The value of $N_s(m, n)$ is obtained explicitly in terms of the quantities

$$\alpha=\alpha_1\cdots\alpha_s,\ \beta=\frac{\beta_1^2}{\alpha_1}+\cdots+\frac{\beta_s^2}{\alpha_s},\ \gamma=n^2-\beta m$$

and the function $\psi(\alpha)$ defined as $+1$, -1 or 0 according as α is a square, non-square or zero in $\mathrm{GF}(p^r)$. It is also proved that $N_s(m, n)\geqq 1$ for $s\geqq 4$ and all cases in which $N_s(m, n)=0$ are enumerated. *L. Carlitz.*

Xiroudakis, Georges. Surd es systèmes multigrades. Mathesis 65 (1956), 371–378.
Gloden, A. Formation de chaînes trigrades avec cinq termes par maillon. Mathesis 65 (1956), 412–414.
Venkatachalam Iyer, R. Analyse multigrade. Mathesis 65 (1956), 416–417.

Additional examples of relations involving equal sums of like powers of integers mentioned MR **8**, 441 and **16**, 1089. *I. Niven* (Eugene, Ore.).

Cugiani, M. Sulla estensione ai polinomi di un teorema di Sylvester-Schur-Erdös. Riv. Mat. Univ. Parma 6 (1955), 261–268.

The author proves among others the following theorems. 1) Let $f(x)$ be an irreducible polynomial of degree n and δ and ε two arbitrary constants satisfying $0<\delta<1$, $\delta<\varepsilon n$. Denote by $P_{x,y}$ the greatest prime factor of $\prod_{0\leq k\leq y} f(x-k)$. Then for $x\geqq x_0$ and $y_0\leqq y<x^\delta$, $P_{x,y}>(1-\varepsilon)y\log y$. 2) Let η, ε, δ be given satisfying $\eta<1$, $\delta<\frac{1}{2}(1-\eta)n$, then for $x\geqq x_0$, $y_0\leqq y<\exp((\sigma\log x\log\log x)^{\frac{1}{2}})$ we have

$$P_{x,y}>y(\log y+\eta\log\log y).$$

Several corollaries are deduced. *P. Erdös.*

Springer, T. A. Note on quadratic forms over algebraic number fields. Nederl. Akad. Wetensch. Proc. Ser. A. 60=Indag. Math. 19 (1957), 39–43.

In this paper a proof, which uses neither Dirichlet's theorem on the prime ideals in an arithmetical progression

or quaternion algebras, is given of the following fundamental theorem of Hasse: If f is a quadratic form in n variables with coefficients in an algebraic number field K, then $f=0$ has a non-trivial solution in K if it has such a solution in all p-adic fields K_p corresponding to the finite or infinite places p of K. Standard results of class field theory are used. *B. W. Jones* (Boulder, Colo.).

Carlitz, L. A note on representations of quadratic forms. Portugal. Math. **15** (1956), 79–81.

This paper gives an elementary simple proof of the following result. If I_k is the identity matrix with k rows, $N_{s-1}(m)$ is the number of solutions of $\sum_{i=1}^{s-1} u_i^2=m$ and B is the matrix $\begin{bmatrix} m & 0 \\ 0 & I_r \end{bmatrix}$ of order $r+1$, then the number of solutions of $UI_sU'=B$ in s by r matrices U with integer elements is

$$2^r s(s-1)\cdots(s-r+1)N_{s-r}(m).$$

This is an extension of a result of Siegel for the case $s=5$, $r=1$. *B. W. Jones* (Boulder, Colo.).

Linnik, Y. V. An application of the theory of matrices and of Lobatschevskian geometry to the theory of Dirichlet's real characters. J. Indian Math. Soc. (N.S.) **20** (1956), 37–45.

This paper contains a brief exposition of the author's recent work [Vestnik Leningrad Univ. **10** (1955), no. 2, 3–23, no. 5, 3–32, no. 8, 15–27; MR **18**, 193] on the distribution of the reduced forms $ax^2+2bxy+cy^2$ with integral a, b, c and $ac-b^2=D>0$, as $D\to\infty$. The connection between this work and the problem of the existence of zeros of real Dirichlet L-functions near $s=1$ is indicated, and some new results are mentioned.

H. Davenport (London).

Inkeri, K. Über eine Verallgemeinerung des letzten Fermatschen Satzes. Ann. Univ. Turku. Ser. A. **23** (1956), 16 pp.

Extending a result of J. L. Selfridge, C. A. Nicol and H. S. Vandiver [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 970–973; MR **17**, 348], the author proves that, if l is a prime, $2<l\leqq 4001$, and α, β, γ are real integers of the field of the primitive lth roots of unity, such that $\alpha\beta\gamma\neq 0$, then $\alpha^l+\beta^l+\gamma^l\neq 0$. *T. Estermann* (London).

Shao Pin-Tsung. On the distribution of the values of a class of arithmetical functions. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 569–572.

This note announces some results on the distribution of values of certain classes of arithmetic functions. One class includes $\varphi(n)$ and $\sigma(n)$ and its definition is suggested by properties of these two functions. The results generalize those of Schinzel and Sierpiński [Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 463–466, 467–469; MR **16**, 675] and include more recent results obtained independently by Schinzel and Wang [Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 207–209; MR **18**, 17]. Proofs are to appear in Acta Scientiarum Naturalium Universitatis Pekinensis.

R. D. James (Vancouver, B.C.).

See also: Sierpiński, p. 711; Khan, p. 713; Brauer, p. 716; Lustig, p. 719.

Analytic Theory of Numbers

Tchudakoff, N. G. Theory of the characters of number semigroups. J. Indian Math. Soc. (N.S.) **20** (1956), 11–15.

This paper was communicated by title to the International Colloquium on Zeta-functions held at Bombay, 1956. It seems to contain a brief report on the results obtained by the author and others in several Russian papers (for the latter, some seven titles are given). Since the present paper is somewhat condensed, the reviewer was not able to extract from it an exact statement of all of its results. However, it may be helpful to the reader of these reviews if one of the theorems is reproduced here, in order that he will get an idea of what kind of result the author has proved: Let $\mathfrak{G}$ be some semigroup of entire ideals of the algebraic number field of degree n; let $\chi(\mathfrak{a})$ be a character of an ideal $\mathfrak{a}$ with $|\chi(\mathfrak{a})|=1$. If $\mathfrak{G}$ has a basis consisting of N prime ideals, where $n<N$, then

$$\sum_{1\leqq N(\mathfrak{a})\leqq e^x} \chi(\mathfrak{a})=\Omega(\sqrt{\lg_3 x}).$$

If $\mathfrak{G}$ has a basis consisting of an infinity of prime ideals and the number of prime ideals $\mathfrak{p}\in\mathfrak{G}$ with $N(\mathfrak{p})\leqq x$ is equal to $\pi(x)=O(\lg_4 x)$, then

$$\sum_{1\leqq N(\mathfrak{a})\leqq e^x} \chi(\mathfrak{a})=\Omega((\lg_3 x)^\mu \ (0<\mu<\tfrac{1}{2}).$$

P. Roquette (Hamburg).

Deuring, M. The zeta-functions of algebraic curves and varieties. J. Indian Math. Soc. (N.S.) **20** (1956), 89–101.

This paper is expository in character. The author reviews in some detail the conjectures (and proofs for some special cases beyond the definite conclusions of Hasse and Weil for function fields of one variable over a finite field) on the zeta functions of algebraic varieties of finite Kroneckerian dimension. The bibliography of the paper includes the important work of Shimura, Taniyama and Weil on complex multiplications; Lang's recent results on the class field theory of algebraic varieties are not included. *O. F. G. Schilling* (Chicago, Ill.).

★ **Yamamoto, Koichi. Theory of arithmetic linear transformations and its application to an elementary proof of Dirichlet's theorem about the primes in an arithmetic progression.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 266–267. Science Council of Japan, Tokyo, 1956.

This article has appeared previously in greater detail [J. Math. Soc. Japan **7** (1955), 424–434; MR **18**, 18]. No bibliography is given in this shortened version.

Lustig, Gerhard. Über die Zetafunktion einer arithmetischen Mannigfaltigkeit. Math. Nachr. **14** (1955), 309–330 (1956).

The results of this paper are related to some of the work of Lang and Weil on the radius of convergence for the zeta function of absolutely irreducible varieties without singularities over a finite field [Amer. J. Math. **76** (1954), 819–827; MR **16**, 398]. The author considers zeta functions of rings in fields of Kroneckerian dimension 2 which are determined by consistent collections of local rings (which are nonsingular in the sense of Krull). The latter assumption permits the enumeration of the number of residue classes of an ideal in a component local ring in terms of the

number of elements of the residue class field belonging to the associated maximal ideal (part I). A reduction to dimension 1 is used to show that his functions converge absolutely if $\mathrm{Re}(s)\geqq 2$.

O. F. G. Schilling.

Delange, Hubert. Sur la distribution des entiers ayant certaines propriétés. Ann. Sci. Ecole Norm. Sup. (3) 73 (1956), 15–74.

This paper is concerned with a number of problems whose type is best stated as finding an equivalent expression for the number of numbers not exceeding x for $x\to\infty$, and possessing one or more given properties. The best known example is the prime number theorem, which asserts that the number of primes not exceeding x is asymptotic to $x/\log x$. The author has generalized a Tauberian theorem appropriate for the proof of the prime number theorem [cf. D. V. Widder, The Laplace transform, Princeton, 1942; MR 3, 232] in a series of papers [C. R. Acad. Sci. Paris 232 (1951), 465–467, 589–591; Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 213–242; MR 12, 405, 497; **16**, 921].

Some of his generalized theorems were applied to prove certain asymptotic formulae in the theory of numbers related to $\omega(n)$ and $\Omega(n)$, which denote, respectively, the number of distinct prime factors, and the total number of primes in the positive integer n [C. R. Acad. Sci. Paris 232 (1951), 1392–1393; Mém. Soc. Roy. Sci. Liége (4) **16** (1955), no. 1–2; Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 87–103; MR **12**, 677; **17**, 965]. The present paper gives a complete proof for a comprehensive group of problems, including many interesting new results. A first group of theorems imposes various conditions on integers such as $\omega(n)=q$, and $\Omega(n)\equiv r \pmod{q'}$ when q and $q'>1$ and r are integers, and n may be also "quadratfrei". A second group deals with integers belonging to an arithmetical progression while $\omega(n)$ and $\Omega(n)$ are restricted as above. For example, the number of numbers not exceeding x such that $(k, l)=1$, $n\equiv l \pmod k$ and $\Omega(n)\equiv r \pmod q$ tends to x/kq as $x\to\infty$. A particular case of $q=2$ is found in Landau, Handbuch der Primzahlen [Bd. 2, 2nd ed., Chelsea, New York, 1953, p. 630; MR **16**, 904]. A third group is studied on certain set E of prime numbers while primes for integers n in $\omega(n)$ and $\Omega(n)$ are restricted to this set. And finally, further generalizations are given by limiting integers to a form $n=p_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}m$, where primes p's belong to the set E and no p's divide m.

S. Ikehara (Tokyo).

Newman, Morris. Some theorems about $p_r(n)$. Canad. J. Math. 9 (1957), 68–70.

Let $p_r(n)$ be the coefficient of x^n in $\prod_{k=1}^{\infty}(1-x^k)^r$, if n is a non-negative integer, and 0 otherwise. The author proves: A. Let $r=4, 6, 8, 10, 14, 26$, let p be a prime >3 such that $r(p+1)\equiv 0 \pmod{24}$, and let $\Delta=r(p^2-1)/24$. Then if $R\equiv r \pmod p$ and $n\equiv\Delta \pmod p$, we have $p_R(n)\equiv 0 \pmod p$. This result generalizes the Ramanujan partition congruences for the moduli 5, 7, 11. B. Let $r=2, 4, 6, 8, 10, 14, 26$. Then $p_r(n)$ vanishes for arbitrarily long strings of consecutive values of n, arbitrarily many in number. Both results follow from a congruence relation proved by the author in an earlier paper [J. London Math. Soc. **30** (1955), 488–493; MR **17**, 15]. He states without proof five other relations of which $p_6(3n+2)=9p_6(\frac{1}{3}n)$ is a typical example.

H. D. Kloosterman (Leiden).

★ **Tatuzawa, Tikao. Additive prime number theory in the totally real algebraic number field.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 261–263. Science Council of Japan, Tokyo, 1956.

This article has appeared previously in greater detail [J. Math. Soc. Japan 7 (1955), 409–423; MR 18, 113].

Korobov, N. M. On completely uniform distribution and conjunctly normal numbers. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 649–660. (Russian)

For a certain strictly increasing sequence of positive integers $\{n_\nu\}$ ($\nu=1, 2, 3, \cdots$) and an integer $q>1$, the function $\alpha(x)$ is defined by

$$\alpha(x)=\sum_{\nu=1}^{\infty}\left(\frac{1}{q^{n_\nu}}-\frac{1}{q^{n_{\nu+1}}}\right)\frac{x^\nu}{p^2{}_\nu},$$

where $\{p_\nu\}$ is a strictly increasing sequence of primes. It is proved that the fractional parts of the function $\varphi(x)=\alpha(x)q^x$ are completely uniformly distributed. It is also shown that

$$\sum_{x=1}^{P} e^{2\pi i\varphi(x)}=O(P^{\frac12}\log P),$$

$$\sum_{x=1}^{P} e^{2\pi iF_s(x)}=O(P^{\frac12}\log P,$$

where

$$F_s(x)=m_1\varphi(x+1)+m_2\varphi(x+2)+\cdots+m_s\varphi(x+s),$$

and $m_1, m_2, \cdots, m_s$ are arbitrary integers not all zero. These theorems extend and improve earlier results of the author [Izv. Akad. Nauk SSSR. Ser. Mat. **14** (1950), 215–238; MR 12, 321].

It is also proved that if $\varphi(x)$ is any completely uniformly distributed function, and $q_1, q_2, \cdots, q_s$ are integers greater than unity, and $\alpha_1, \alpha_2, \cdots, \alpha_s$ are defined by

$$\alpha_\nu=\sum_{k=1}^{\infty}\frac{[\{\varphi(sk+\nu)\}q_\nu]}{q_\nu{}^k}\qquad(1\leqq\nu\leqq s)$$

then the system of functions $\alpha_1q_1{}^x, \cdots, \alpha_sq_s{}^x$ is uniformly distributed in s-dimensional space. Here brackets $[\cdots]$ and $\{\cdots\}$ denote integral and fractional parts respectively. This result was stated earlier by the author without proof [Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 13–16; MR **14**, 144].

R. A. Rankin (Glasgow).

See also: Cugiani, p. 718; Linnik, p. 719.

Theory of Algebraic Numbers

★ **Weil, André. On a certain type of characters of the idèle-class group of an algebraic number-field.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 1–7. Science Council of Japan, Tokyo, 1956.

Let k be a finite algebraic number field, I the idèle group of k and P the principal idèle group of k. The author introduces here certain new characters of the idèle class group I/P as yet unknown in arithmetic and discusses some of their properties.

Let I_0 and I_∞ denote the non-archimedean and archimedean parts of I, respectively, so that $I=I_0\times I_\infty$, $I_\infty=\prod k_i^*$, where k_i ($1\leqq i\leqq r$) are the archimedean completions of k and are identified with either the real field R or the complex field C. Let χ be a character of

I/P, i.e. a continuous homomorphism of I into C^* such that $\chi(P)=1$, and let χ_∞ be the restriction of χ on I_∞. The character χ is then called of type (A_0) if the character $X=\chi_\infty^{-1}$ on I_∞ is of the form $X(a)=\pm\prod a_i^{s_i}\bar{a}_i^{t_i}$ $(a=(a_i)$, $a_i\in k_i^*)$ with some integers s_i and t_i. Let $\tilde{\chi}$ be the "Grössencharakter" of k attached canonically to χ. The fact $\chi(P)=1$ then implies a certain relation between $\tilde{\chi}$ and X, and it follows that if χ is of type (A_0), the values of $\tilde{\chi}$ are contained in some finite algebraic number field K. (More generally, the author defines characters of type (A) for which the values of $\tilde{\chi}$ are always algebraic numbers. But the definition of such characters is omitted here.)

Now, let $\mathfrak{p}$ be any prime ideal of K and p the rational prime divisible by $\mathfrak{p}$. Furthermore, let $\mathfrak{f}$ be the conductor of χ and I' the subgroup of all $\chi=(\chi_q)$ in I_0 such that $\chi_q=1$ for every q dividing $p\mathfrak{f}$. By the choice of K, the restriction χ' of χ on I' takes values in K and, hence, in $K_\mathfrak{p}$, the $\mathfrak{p}$-completion of K. Thus, χ' can be uniquely extended to a continuous homomorphism $\chi_\mathfrak{p}$ of I into $K_\mathfrak{p}^*$ such that $\chi_\mathfrak{p}(P)=1$. If ω is any character of K^*, $\omega\circ\chi_\mathfrak{p}$ then gives a character of finite order on I/P and it defines, by class field theory, a finite cyclic extension of k. The compositum of all such extensions, for all possible $\mathfrak{p}$ and ω, then gives rise to an abelian extension $k(\chi)$ of k, canonically associated with the character χ. It is noticed that if χ is a character of finite order (and, hence, is of type (A_0)), $k(\chi)$ is nothing but the finite cyclic estension of k associated with χ by class field theory, and that for a suitable χ, $k(\chi)$ gives the maximal cyclotomic extension over k.

Finally, the author emphasizes the importance of such characters of type (A_0), pointing out that there are interesting relations, as shown by Taniyama, between those characters χ and their fields $k(\chi)$ on one hand and abelian varieties with complex multiplication and their zeta-functions on the other. *K. Iwasawa.*

★ **Kawada, Yukiyosi. Some remarks on class formations.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 239–241. Science Council of Japan, Tokyo, 1956.

This article has appeared previously in greater detail [J. Math. Soc. Japan **7** (1955), 453–490; MR **18**, 114]. However, two additional references are cited in the current version [E. Witt, J. Reine Angew. Math. **176** (1937), 126–140; Y. Kawada, Sûgaku **6** (1954), 129–150 (in Japanese)].

See also: Moriya, p. 714; Reiner, p. 717; Springer, p. 718; Tchudakoff, p. 719; Lustig, p. 719.

Geometry of Numbers

Few, L. Covering space by spheres. Mathematika **3** (1956), 136–139.

If Λ is a lattice in three-dimensional space with the property that the spheres of radius 1 centered at the lattice points cover the whole of space, then

$$d(\Lambda)\leq\frac{32}{5\sqrt{5}}$$

as was proved by Bambah [Proc. Cambridge Philos. Soc. **50** (1954), 203–208; MR **15**, 780] and later by Barnes [Canad. J. Math. **8** (1956), 293–304; MR **17**, 1060]. The author gives a third proof which is fully elementary and which, unlike the two former proofs, makes no use of the theory of reduction of ternary quadratic forms.

J. F. Koksma (Amsterdam).

Dupač, Václav. On a stochastic modification of a problem in geometry of numbers. Czechoslovak Math. J. **5(80)** (1955), 492–502. (Russian. English summary)

Let J be a closed curve in the u, v plane defined in polar coordinates by $r=g(\theta)$, where $g(\theta)$ is positive, periodic with period 2π, and satisfies a Lipschitz condition. Suppose further that J contains an arc $v=f(u)$, for some interval of u in which $k_0<|f'(u)|<k_1$ and $|f''(u)|<k_2$ for suitable positive constants. Let $J(x)$ denote the magnification of J given by $r=x^{\frac{1}{2}}g(\theta)$, where $x>0$. Let the unit squares of the plane be enumerated in a fixed order as $Q_1, Q_2, \cdots$, and let $Z_1, Z_2, \cdots$ be independent random points, uniformly distributed in $Q_1, Q_2, \cdots$ respectively. Let $A(x)$ denote the number of those points $Z_1, Z_2, \cdots$ which lie inside $J(x)$ and put $R(x)=A(x)-Px$, where P is the area inside J. The main results are (1) the proposition $R(x)\neq o(x^{\frac{1}{4}}\log^{\frac{1}{4}}x)$ has probability 1; (2) if $x_n>n^\delta$ for some fixed $\delta>0$ then the proposition $R(x_n)=O(x_n^{\frac{1}{4}}\log^{\frac{1}{4}}x_n)$ has probability 1. A weaker result for more general regions is also given.

H. Davenport (London).

ANALYSIS

Functions of Real Variables

Četković, Simon. Inexactitude de quelques propositions généralisées des accroissement finis. Bull. Soc. Math. Phys. Serbie **7** (1955), 119–124. (Serbo-Croatian. French summary)

For any continuous function $f(x)$ having a left-hand derivative $f_-'(x)$ and a right-hand derivative $f_+'(x)$ at each interior point of its interval of definition, a generalization of the law of the mean has been given J. Karamata [Srpska Akad. Nauka. Zb. Rad. **7**, Mat. Inst. **1** (1951), 119–124; MR **13**, 329], and this result has been extended by V. Vučković [same Zb. **18**, Mat. Inst. **2** (1952), 159–166; MR **14**, 625]. The author now gives counterexamples showing that these results, as stated, are false.

Actually, as W. R. Utz wrote in a letter to the reviewer, dated December 16, 1953, the results become valid when, in their statements and in the statement of the generalization of Rolle's theorem on which they are based, the inequalities $p>0$, $q>0$ are replaced by $p\geq0$, $q\geq0$. Thus, for example, the generalization of Rolle's theorem should read: If $f(x)$ is continuous in the closed and bounded interval $a\leq x\leq b$ and satisfies $f(a)=f(b)=0$, and if $f_-'(x)$ and $f_+'(x)$ exist at each point of the open interval $a<x<b$, then there exist values p, q, ξ, with $a<\xi<b$, $p\geq0$, $q\geq0$, $p+q=1$, such that $pf_+'(\xi)+qf_-'(\xi)=0$.

E. F. Beckenbach (Los Angeles, Calif.).

Tăutu, Dana. Sur la représentation de la variation totale d'une fonction continue par une intégrale riemannienne. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 59–66. (Romanian. Russian and French summaries)

Soit $N_f(t)$ le nombre des solutions de l'equation $f(x)=t$ où $f(x)$ est une fonction réelle de variable réelle $\cdots$. Une condition nécessaire et suffisante pour que la fonction $N_f(t)$, où $f(x)$ est continue, soit intégrable Riemann, au

sens généralisé, est que $mf(M')=0$, où m est la mesure de Lebesgue, M l'ensemble des points d'extremum de $f(x)$, M' est l'ensemble dérivé de M. (From the author's summary.)

The author defines the generalized sense of Riemann integration and points out the connection between his result and the theorem of Banach (Fund. Math. **8** (1925) 225–236) that a continuous $f(x)$ is of bounded variation if and only if $N_f(t)$ is Lebesgue integrable.

Pollak, H. O. A remark on "Elementary inequalities for Mills' ratio" by Yûsaku Komatu. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1956), 110.

It is proved that the best possible upper bound of the form $2/(\sqrt{(x^2+2b)}+x)$ for Mills' ratio $e^{x^2/2}\int_x^\infty e^{-t^2/2}dt$ is given by $b=4/\pi$.
Y. Komatu (Tokyo).

★ **Гарди, Г. Г.; Литтльвуд, Дж. Е.; и Полиа, Г. [Gardi G. G. (Hardy, G. H.); Littl'vud, Dž. É. (Littlewood, J. E.); i Polia (Pólya) G.] Неравенства. [Inequalities].** Gosudarstv. Izdat. Inostr. Lit., Moscow, 1948. 456 pp. 25.60 rubles.

This is a translation by V. I. Levin of the edition of 1934, with 81 pages of appendices by Levin and Stečkin. The appendices contain further inequalities, mostly taken from the literature since 1934, but some of them apparently published here for the first time. The following specimens are numbered according to the appendices in which they appear. (1) Some general integral inequalities for convex functions which imply, for example,

$$\frac{1}{n}\sum_1^n \phi\left(\frac{k}{n+1}\right) \geqq \frac{1}{n-1}\sum_1^n \phi\left(\frac{k}{n}\right)$$

if ϕ is continuous and convex on $(0, 1)$; if $p(x)$ is nondecreasing on $(0, \frac{1}{2})$ and $p(x)=p(1-x)$ then

$$\int_0^1 p(x)\phi(x)dx \leqq \int_0^1 p(x)dx \int_0^1 \phi(x)dx$$

for every continuous convex ϕ. (2) Carlson's inequality and its generalizations, including a more general form of Levin's inequality [Mat. Sb. N.S. **3(45)** (1938), 341–345]. (5) Numerous analogues of Theorem 258 (Wirtinger) are deduced from the following general theorem. Let $f(x)$ have period 2π, $\int_0^{2\pi} f(x)dx=0$, with an absolutely continuous $(k-1)$th derivative; let $J_r(f)=(\int_0^{2\pi}|f|^r dx)^{1/r}$. If $1\leqq s\leqq r\leqq\infty$, $\mu^{-1}=1+r^{-1}-s^{-1}$, then

$$J_r(f)\leqq C_{k,\mu}J_s(f^{(k)}),\quad C_{k,\mu}=\min_\xi J_\mu(\phi_k-\xi),$$

$$\phi_k(t)=\pi^{-1}\sum_{n=1}^\infty n^{-k}\cos(nt-\tfrac{1}{2}k\pi);$$

the constant is exact when $r=\infty$. (11) Sharpening of Theorem 338: If $0<p<1$, $0<r\leqq p$, $a\geqq 0$ and not all a_n are 0, and $r_n=\sum_0^\infty a_k$, then

$$(*)\qquad \sum_{n=1}^\infty{}' n^{-r}s_n^p > \left(\frac{p}{1-r}\right)^p \sum n^{-r}(na_n)^p,$$

where in $\sum'$ the first term is to be multiplied by $(1+p/(1-r))$; the constant $[p/(1-r)]^p$ is exact. (12) The constant in Theorem 345 is shown not to be exact, and the exact constant is determined for $0<p\leqq\frac{1}{2}$ (it is $[p/(1-p)]^p$). A more general inequality is proved ((*) without the prime and with a different constant). (13) Extension of Theorem 347. If $0<p<1$, $f\geqq 0$, $F(x)=\int_x^\infty F(u)du$, then

$$\int_1^\infty x^{-1}(xf)^p dx < \left(\frac{1-p}{p}\right)^p \int_1^\infty x^{-1}(\log x)^{-p}F^p dx;$$

a more general result is given with iterated logarithms. There is a discrete analogue. (14) The best constant is found for the 2-parameter Hilbert inequality (Theorem 339): $K(p, q)=\{\pi\csc(\pi/\lambda p')\}^\lambda$. (16) If $1<p<\infty$, $f\geqq 0$, $f\not\equiv 0$, $f\in L^p(-\infty, \infty)$, $k>0$, and $F(x)=e^{kx}\int_x^\infty e^{-ku}f(u)du$, then $\int_{-\infty}^\infty F^p dx<k^{-p}\int_{-\infty}^\infty f^p dx$ (Stepanov).
R. P. Boas, Jr. (Evanston, Ill.).

Ghizzetti, Aldo. Sulla convergenza dei procedimenti di calcolo, degli integrali definiti, forniti dalle formule di quadratura. Rend. Sem. Mat. Univ. Padova **26** (1956), 201–222.

Massalski, J. Zips. Math. Gaz. **40** (1956), 267–268.

If the function f is defined in $[a, b]$ and the number n is given, then the author sets

$${}_n\overset{b}{\underset{a}{Z}} f=\lim_{N\to\infty}\sum_{\nu=0}^N(-1)^\nu\binom{n}{\nu}f(b-\nu h)h^{-n},$$

where $h=(b-a)/N$, provided that this limit exists, and he calls it "zip" of the rank n of f in $[a, b]$. For $n=-1$ this gives $\int_a^b f(x)dx$ and, if for a positive integer n the function f possesses an nth derivative in $[a, b]$, then

$${}_n\overset{b}{\underset{a}{Z}} f=f^{(n)}(b).$$

A. Rosenthal (Lafayette, Ind.).

Leach, E. B. Functions with preassigned derivatives. Amer. Math. Monthly **63** (1956), 653–655.

The author generalizes E. Borel's theorem [Ann. Sci. Ecole Norm. Sup. (3) **12** (1895), 9–55, pp. 35–44] on the existence of a function of one real variable x whose derivatives of all orders have preassigned values at a point, to the case where these given values of the derivatives depend on m parameters; namely he proves the theorem: Let $\{a_k(u)\}$ $(k=0, 1, 2, \cdots)$ be a sequence of C^∞ functions of the m-dim. real variable $u=(u_1, \cdots, u_m)$, each defined for $u\in U$, an open set in R^m. Then there is a C^∞ function $f(x, u)$ defined for $x\in R_1$ and $u\in U$, such that $\partial^k f(0, u)/\partial x^k=a_k(u)$ $(k=0, 1, 2, \cdots)$. Then (by induction on n) he extends this theorem to the case of n real variables, i.e. to the case of an infinitely differentiable function $f(x, u)$ where $x=(x_1, \cdots, x_n)$ and the partial derivatives of f with respect to the $x_1, \cdots, x_n$ at $x=0$ are preassigned C^∞ functions of u. {Reviewer's remark: An extension of Borel's theorem to n variables [but without dependence on parameters] was also given by H. Mirkil [Proc. Amer. Math. Soc. **7** (1956), 650–652; MR **18**, 23]; the papers of the two authors are independent of each other.}
A. Rosenthal (Lafayette, Ind.).

Crum, M. M. On positive-definite functions. Proc. London Math. Soc. (3) **6** (1956), 548–560.

The real or complex-valued function $p(x)$ of

$$x=(x^1, x^2, \cdots, x^m)$$

is called "positive-definite" if the Hermitean form

$$\sum_{\mu,\nu=1}^N\sum p(x_\mu-x_\nu)\rho_\mu\bar\rho_\nu\geqq 0$$

for all finite sets x_μ and all complex ρ_μ. As S. Bochner [Math. Ann. **108** (1933), 378–410] first proved, one has for continuous $p(x)$ and $m=1$

$$(*)\qquad p(x)=\int_{-\infty}^{+\infty}e^{ix\xi}dV(\xi),$$

where $V(\xi)$ is non-decreasing and bounded. Then F. Riesz [Acta Litt. Sci. Szeged 6 (1933), 184–198] showed that, if $p(x)$ is measurable, (*) holds for almost all x. Moreover, I. J. Schoenberg [Ann. of Math. (2) 39 (1938), 811–841; Duke Math. J. 9 (142), 96–108; MR 3, 232 gave an extension of Bochner's theorem to R^m. The author now proves the following theorem: If $p(x)$ is positive-definite and measurable, then $p(x)=q(x)+r(x)$ where (1) $q(x)=(R^m)\int e^{i(x,\xi)}d\mu(\xi)$ with $(x,\xi)=\sum_{i=1}^m x^i\xi^i$, (2) $r(x)=0$ for almost all x, (3) both $q(x)$ and $r(x)$ are positive-definite. The principal new result of the author is the fact that $r(x)$ is also positive-definite. By an example using Hamel's basis it is shown by him that the assumption of the measurability of $p(x)$ is essential. Part (2) is sharpened if $m>1$ and $p(x)$ is "isotropic", i.e. $p(x)=f(|x|)$; namely then $r(x)=0$ for $x\neq 0$. For the case of positive-definite functions on the surface of a sphere a corresponding theorem is proved (even in a simpler manner).

A. Rosenthal (Lafayette, Ind.).

Reichelderfer, Paul V. A covering theorem for transformations. Math. Japon. 4 (1956), 13–19.

Let T be a continuous bounded transformation from a bounded domain D in Euclidean n-space R^n into R^n. The covering theorem established in this paper involves many concepts whose explicit definitions may be found in the book by T. Rado and P. Reichelderfer, "Continuous transformations in analysis" [Springer, Berlin, 1955; this book will be referred to as CTA; MR 18, 115]. Let D be a domain whose closure is contained in D. Then D is termed a model domain for (T, D) if there exists a domain Δ in R^n such that D is a component of $T^{-1}\Delta$. If this is the case, then it follows that the topological index $\mu(T, x, D)$ [see CTA, II.2.2] has a constant value for $x \in TD$. This constant value will be denoted by $i(T, D)$. A domain D is termed a model indicator domains for T (briefly, an m.i.d. T) if (i) TD is a bounded domain, (ii) D is a component of $T^{-1}TD$, (iii) the closure of D is contained in D, (iv) $i(T, D)\neq 0$. Extending the concept of "covering in the Vitali sense" [see S. Saks, Theory of the integral, 2nd ed., Warszawa, 1937, ch. IV, § 3], the author sets up the following definition. A family F of subsets of D covers a set $U \subset D$ in the T sense of Vitali if for every point u of U there exists a sequence of sets X_j in F such $u \in X_j$ and TX_j is a regular sequence of sets tending to Tu. In terms of these concepts and definitions, the author establishes the following covering theorem. Let F_0 be a family of domains D such that each $D \in F_0$ is an m.i.d. T and for each $D \in F_0$ the Lebesgue measure of the image of the boundary of D is equal to zero. Let U_0 be a non-empty subset of D which is covered by F_0 in the T sense of Vitali. Then there exist pair-wise disjoint sets U, U', U'' whose union is U_0 and a sequence of pair-wise disjoint domains D_k in F_0 satisfying the following conditions. 1) $TU \subset X_\infty$, where X_∞ is the set of those points x where the multiplicity function $k(x, T, D)$ (see CTA, II.3.3) is equal to ∞. 2) $U' \subset UD_k$. 3) The Lebesgue measure of TU'' is equal to zero. Furthermore, if U_0 is a Borel set, then U, U', U'' are also Borel sets. If T is essentially of bounded variation in D, then the Lebesgue measure of X_∞ is equal to zero [see CTA, II.3.4, IV.4.1], and the author states the simplified form of his covering theorem for this case.

T. Radó (Columbus, Ohio).

Kovan'ko, A. S. On some new methods and formulas in analysis. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Fak. 1 (1941), no. 1, 47–48. (Russian)

Let $\phi(x, y)$ have continuous second derivatives ϕ_{xy}'' and ϕ_{yx}'', so that, as is well known, we have $\phi_{xy}''=\phi_{yx}''$. Let $\Delta_x\phi=\phi(x+\Delta x, y)-\phi(x, y)$, $\Delta_y\phi=\phi(x, y+\Delta y)-\phi(x, y)$, and $\Delta_{xy}\phi=\Delta_{yx}\phi=\Delta_x(\Delta_y\phi)=\Delta_y(\Delta_x\phi)$. By means of two successive applications of the formula of Lagrange, we obtain

$$(*)\qquad \Delta_{xy}\phi=\phi_{xy}''(x+\theta_1\Delta x, y+\theta_2\Delta y)\Delta x\Delta y,$$

with $0<\theta_j<1$, $j=1, 2$.

The author points out that if $F(x, y)$ and $\Phi(x, y)$ satisfy the same conditions that have been specified for $\phi(x, y)$, and in addition we have $\Phi_{xy}''\neq 0$, then an application of (*) to the function $F+\lambda\Phi$, where the constant λ is chosen to satisfy $\lambda=-\Delta_{xy}F/\Delta_{xy}\Phi$ at any given point (x, y), yields the known extended mean-value result that

$$\frac{\Delta_{xy}F(x,y)}{\Delta_{xy}\Phi(x,y)}=\frac{F_{xy}''(x+\theta_1\Delta x, y+\theta_2\Delta y)}{\Phi_{xy}''(x+\theta_1\Delta x, y+\theta_2\Delta y)}.$$

He notes further that an application of the same device to the mean-value theorem for integrals,

$$\int_V\phi(P)dP=|V|\phi(P^*),$$

where P^* is some point of V, yields the familiar extended mean-values result that if $\Phi(P)$ does not vanish then

$$\frac{\int_V F(P)dP}{\int_V \Phi(P)dP}=\frac{F(P^*)}{\Phi(P^*)};$$

and from this, with $F\Phi$ in place of F, he obtains the well-known result that

$$\int_V F(P)\Phi(P)dP=F(P^*)\int_V\Phi(P)dP.$$

E. F. Beckenbach (Los Angeles, Calif.).

Nikol'skiĭ, S. M. Boundary properties of functions defined in a region with angular points. I. Mat. Sb. N.S. 40(82) (1956), 303–318. (Russian)

The author is concerned with functions f in the class $H_p^{(r)}(G)$ ($r>0$, $1\leq p\leq\infty$, G a plane region) and the corresponding functions ϕ induced on the boundary Γ of G. Inequalities between norms of L^p type for f and ϕ, previously derived for the case of smooth Γ, are here extended to piecewise smooth Γ.

M. G. Arsove.

Gonzalez-Gallarza, Felix Llorente. Reduction of the hypotheses in Schwarz's theorem. Euclides, Madrid 15 (1955), 259–264. (Spanish)

A clear presentation of the proof of the theorem of Schwartz, concerning the interchangeability of the order of differentiation in the evaluation of a mixed partial derivative, is given under weakened hypothesis and strengthened conclusion. It is pointed out that the existence and equality of both $\partial^2 f/\partial x\partial y$ and $\partial^2 f/\partial y\partial x$ can be put in the conclusion when the hypothesis contains the assumption of the existence of the limit of the difference quotient

$$[f(x_1+h, y_1+k)-f(x_1y_1+k)-f(x_1+h, y_1)+f(x_1, y_1)]/hk$$

as $(h, k)\to(0, 0)$.

E. F. Beckenbach.

Del Pasqua, Dario. Sui funzionali derivati dei funzionali analitici. Rend. Mat. e Appl. (5) 15 (1956), 211–227.

Soit $F[y]$ une fonctionnelle définie et analytique dans un ouvert $\mathcal{R}$ de l'espace $\mathcal{S}$ des fonctions localement analytiques. Dans la forme initiale de la théorie des fonctionnelles analytiques, la «dérivée d'ordre n» de $F[y]$ est, non pas la fonctionnelle multilinéaire que l'on associe

couramment à cette désignation, mais la fonction indicatrice de cet opérateur. Ce travail propose quelques perfectionnements de la théorie de Fantappiè, en ce qui concerne les domaines de telles fonctions indicatrices et les développements tayloriens des fonctionnelles analytiques. {Note du reviewer — L'auteur semble continuer à ignorer le changement radical et les progrès opérés dans cette théorie [voir les travaux de G. Köthe, A. Grothendieck, C. Silva Dias, H.-G. Tillmann, etc. et du reviewer, sur la mise au point moderne de la théorie des fonctionnelles analytiques]. Après ce changement, tous les efforts de l'auteur risquent de rester inefficaces.}

J. Sebastião e Silva (Lisbonne).

Mlak, W. Note on the mean value theorem. Ann. Polon. Math. 3 (1956), 29–31.

Let E be a linear topological locally convex space and let A be a closed, convex subset of E. Let E^* designate the class of all linear (additive and continuous) functionals defined on E. Moreover, by $\varphi(t)$ denote a real valued, continuous, increasing function in a given interval Δ, while $x(t)$, defined on Δ with values in E, is assumed to be weakly continuous in Δ. The author proves the following generalization of the mean value theorem: For every $f \in E^*$ there is an at most denumerable set $\Delta_f C \Delta$ such that for every $t \in \Delta - \Delta_f$ there exists a sequence $y_n \in A$ and a sequence of reals $\tau_n \to 0^+$ such that

$$f\{[x(t+\tau_n)-x(t)]/[\varphi(t+\tau_n)-\varphi(t)]-y_n\} \to 0.$$

Then for $t_1, t_2 \in \Delta$ and $t_1 \neq t_2$ one has

$$[x(t_1)-x(t_2)]/[\varphi(t_1)-\varphi(t_2)] \in A.$$

In a second theorem $\varphi(t)$ is replaced by t and it is supposed that for every $f \in E^*$ the function $f[x(t)]$ is absolutely continuous in Δ. *A. Rosenthal* (Lafayette, Ind.).

See also: Riesz and Sz.-Nagy, p. 747; Mazurkiewicz, p. 768.

Measure, Integration

Goetz, A.; Hartman, S.; and Steinhaus, H. Invariant measures in spaces with a transitive group of transformations. Prace Mat. 2 (1956), 139–145. (Polish. Russian and English summaries)

The authors prove: For a real function $f(x, y)$ defined in the square ($0 \leq x \leq 1$, $0 \leq y \leq 1$) and for any two measurable sets A, B on a sphere with the area 1, there exists a rotation τ with $m(A \cap \tau B) = f[m(A), m(B)]$ if and only if $f(x, y) = x \cdot y$. They derive the "if"-part from the following proposition: If H is a closed subgroup of a compact group G and μ the normed Haar measure on G, m the normed invariant measure on G/H, then

$$\int_G m(A\tau \cap B)\mu(d\tau) = m(A)m(B)$$

for any Borel sets $A, BCG/H$.

They also prove the existence and uniqueness of a Borel measure invariant under a transitive group G of equi-continuous mappings of a compactum E onto E. [For more general results see L. H. Loomis, Duke Math. J. 16 (1949), 193–208; MR 10, 600; I. E. Segal, J. Indian Math. Soc. (N.S.) 13 (1949), 105–130; MR 11, 425.] Their proof is based on the establishment of a homeomorphism between E and G/H, where H is the subgroup of those elements of G which leave invariant a fixed point of E. *H. M. Schaerf* (St. Louis, Mo.).

See also: Riesz and Sz.-Nagy, p. 747; Bouligand, Choquet, Kaloujnine et Motchane, p. 758; Mazurkiewicz, p. 768.

Functions of Complex Variables

Grau, A. A.; and Goldbeck, B. T., Jr. Algebraic properties of classes of functions. Amer. Math. Monthly 63 (1956), 636–638.

The authors generalize the notions of odd and even functions in the following manner: Let ω be a primitive nth root of unity. A function $f(z)$ of a complex variable z is called to be of type (n, k) with $0 \leq k < n$ if $f(\omega z) = \omega^k f(z)$. Then the following theorem is proved: Every complex valued function $f(z)$ may be expressed uniquely as

$$f(z) = f_0(z) + f_1(z) + f_2(z) + \cdots + f_{n-1}(z),$$

where $f_k(z)$ is of type (n, k). Moreover, it is proved that (with suitable definitions of the operations) the ring of classes, ${}_n\mathfrak{F}_k$, of the functions of type (n, k) is isomorphic to the ring of residue classes of integers modulo n.

A. Rosenthal (Lafayette, Ind.).

Suvorov, G. D. On the continuity of univalent mappings of arbitrary closed domains. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 777–779. (Russian)

Let D be a plane domain containing $z=0$, let l_ξ, l_η denote respectively the intersections of D with $x=\xi$, $y=\eta$. A real function $f(p)$ defined in D is said to be of class BL in D if $f(p)$ is absolutely continuous on almost all l_x and l_y, and if $(\partial f/\partial x)^2$ and $(\partial f/\partial y)^2$ are summable in D. A mapping $w = T(z) = f_1(x, y) + if_2(x, y)$ is called of class BL if f_1 and f_2 are of class BL; C_k' denotes the class of mappings $T = f_1 + if_2$ such that the mappings

$$T_n(z) = f_{1n} + if_{2n},$$

where $f_{in} = f_i$ if $|f_i| \leq n$ and $f_{in} = 0$ if $|f_i| > n$, belong to class BL and satisfy the inequality

$$\iint_D \frac{\sum_{i=1}^2 \operatorname{grad}^2 f_{in}}{(1+\sum_{i=1}^2 f_{in}^2)^2} dxdy \leq k.$$

The author announces an inequality, too long to be reproduced here, concerning distortion in the spherical metric of functions of the class C_k'. *A. J. Lohwater.*

San Juan, Ricardo. Classes semi-analytiques dans des régions convexes. C. R. Acad. Sci. Paris 244 (1957), 292–294.

Although "semi-analytic classes" are not defined in this note, they appear to be classes of functions analytic in certain unbounded domains and satisfying the type of condition associated with Watson's problem [see Mandelbrojt, Séries adhérentes $\cdots$, Gauthier-Villars, Paris, 1952, especially pp. 46–47; the note under review cannot be read without this book at hand; MR 14, 542]. The author's object seems to be to extend some of his own and Mandelbrojt's results about semi-analytic classes to more general domains than had been considered before.

R. P. Boas, Jr. (Evanston, Ill.).

R.-Salinas, Baltasar. Uniqueness problems in the theory of asymptotic series. Calculation of semi-analytic functions by the algorithms of Borel and Stieltjes. Rev. Acad. Ci. Madrid 50 (1956), 191–227. (Spanish)

Let $f(z)$ have, in an unbounded domain contained in

the angle $|\arg z| < \frac{1}{2}\alpha\pi$ of the Riemann surface of $\log z$, an asymptotic expansion $\sum a_n z^{-n}$ such that

$$|f(z) - \sum_{\nu=0}^{n-1} a_\nu z^{-\nu}| \le m_n |z|^{-n}.$$

The function is said to be semi-analytic if the m_n satisfy the Carleman-Ostrowski condition for Watson's problem. The author studies several problems of uniqueness connected with such asymptotic expansions when the region in question is defined by $\Re(z^{1/\alpha}) > a^{1/\alpha}$. With $f(z)$ associate the function

$$F(t) = \sum a_n t^{\alpha n}/(\alpha n)!.$$

Then under various circumstances the author expresses $f(z)$ in terms of $F(t)$ by

$$f(z) = z^{1/\alpha}\int_0^\infty \exp(-z^{1/\alpha}t)F(t)dt = L(f),$$

say. Then he obtains bounds for the difference

$$|L(F) - \sum_{\nu=0}^{n-1} a_\nu z^{-\nu}|.$$

Next he uses integrals like $L(F)$ to study the question of when an asymptotic expansion in one region gives rise to a similar one in a related region. Several illustrations and examples are constructed, in particular an asymptotic series which has "optimal" asymptotic sums $f(z)$ in various half planes which are not branches of the same function. The author also obtains some formulas for semi-analytic functions in terms of Stieltjes transforms along a curve C.

R. P. Boas, Jr. (Evanston, Ill.).

Rizza, Giovanni Battista. Funzioni regolari nelle algebre di Clifford. Rend. Mat. e Appl. (5) 15 (1956), 53–79.

L'auteur continue ses recherches sur les théories de fonctions analytiques dans des algèbres de dimension finie [Pont. Acad. Sci. Comment. 14 (1950), 169–194; Rend. Mat. e Appl. (5) 11 (1952), 134–155, MR 15, 213, 949]. Il établit maintenant une formule générale de type Cauchy (un peu trop compliquée pour être rapportée ici), pour les fonctions analytiques dans les algèbres de Clifford. Ces algèbres étant (sauf celle des nombres complexes) isomorphes à des sommes directes d'algèbres de quaternions, l'auteur considère, en particulier, le cas des fonctions analytiques dont les composantes quaternioniques sont des fonctions analytiques à droite de variables quaternioniques, et il parvient à des formules intégrales pour les fonctions de plusieurs variables quaternioniques. Dans cette étude, les considérations topologiques jouent un rôle essentiel en profondeur. *J. Sebastião e Silva.*

San Juan, Ricardo. Classes semi-analytiques et sommation de séries potentielles divergentes. C. R. Acad. Sci. Paris 244 (1957), 432–434.

Let σ be the isomorphism between s, the set of power series $\sum a_n z^n$ with $\limsup |a_n|^{1/n} < \infty$, and h, the set of functions holomorphic at 0. Let D be an open domain with point of accumulation at 0 and star-shaped with respect to 0; let $\mathcal{D}$ be the set of domains D. The author considers Problem A: To extend the isomorphism σ between s and h by means of another isomorphism σ_1 between a set $S \supset s$ ($S \not\equiv s$) of series $\sum a_n z^n$ and a set $A \supset h$ ($A \not\equiv h$) of functions $f(z)$, each of which is analytic at least in a domain $D \in \mathcal{D}$. The series $\sum a_n z^n \in S$ is called summable and the homologous function $f(z) \in A$ in the isomorphism σ_1 is called the σ_1-sum of the series $\sum a_n z^n$. Having fixed $\mathcal{D}$, a solution σ_1 of problem A contains another σ_2 if the set S_1 of series summable σ_1 contains the set S_2 of series summable σ_2.

For $R_\alpha \equiv |z^{1/\alpha} - a^{1/\alpha}| < a^{1/\alpha}$ ($a>0$, $0 \le \alpha \le 2$) as the set $\mathcal{D}$, it is not possible to obtain a solution σ_1 of problem A which contains all others. The author is then led to consider problem B: To obtain a solution σ_1 of Problem A such that: (1) If $\sum a_n z^n$ is σ_1 summable, its sum coincides with that which is obtained by any solution whatever σ_2 of A. (2) The set S_1 of series summable σ_1 contains the set $S_1{}^1$ of series summable with any other solution $\sigma_1{}^1$ which satisfies the same condition (1) as σ_1. Problem B has a unique solution for $\mathcal{D} = R_\alpha$. *D. Moskovitz.*

Nisigaki, Hisami; and Takasu, Tsurusaburo. General three-dimensional complex function theory. I. Yokohama Math. J. 3 (1955), 53–126.

This paper covers only one-fourth of the material listed in its table of contents. It gives for three-dimension hypercomplex numbers a theory of analytic functions later extended by T. Takasu to n dimensions [same J. 1 (1953), 131–224; MR 16, 350].

J. A. Ward (Holloman A.F.B., N.M.).

Karadžić, Lazar. Quelques propriétés des fonctions définies par la série de Taylor ou de Dirichlet. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser Mat. Fiz. no. 8 (1956), 12 pp. (Serbo-Croatian summary)

By geometric arguments the author shows that if $a_n/a_{n+1} \to \alpha$, $|\alpha| = 1$, the series $\sum a_n z^n$ converges uniformly on $|z| = 1$ except in neighborhoods of α and its antipodal point β if $a_n \to 0$, and is bounded except in neighborhoods of α and β if $\{a_n\}$ is bounded; and α is a singular point. More generally, for the Dirichlet series $\sum a_n e^{-\lambda_n s}$, if $c = \limsup \lambda_n^{-1} \log |a_n|$ and $(a_{n+1}/a_n)e^{-(\lambda_{n+1}-\lambda_n)c} \to 1$, if $g = \lim (\lambda_{n+1} - \lambda_n)$, $\phi = \lim(\arg a_{n-1} - \arg a_n)$, then the Dirichlet series converges uniformly (or is bounded) on interior segments of the part $0 < \Im(s) < (\pi - \phi)/g$ of the line $\Re(s) = c$, according as $a_n e^{-\lambda_n c} = o(1)$ or $O(1)$; and $c + i(2\pi - \phi)/g$ is a singular point.

R. P. Boas, Jr. (Evanston, Ill.).

Walsh, J. L.; and Evans, J. P. On the location of the zeros of certain orthogonal functions. Proc. Amer. Math. Soc. 7 (1956), 1085–1090.

The orthogonal functions $\phi_n(z)$, $n = 1, 2, \cdots$, studied here are defined relative to a given finite region R of the z plane and a given set $b_1, b_2, \cdots$ of points interior to R, as follows. (1) Each $\phi_n(z)$ is analytic in R; (2) $\phi_n(b_k) = 0$ for $k = 1, 2, \cdots, n-1$, $\phi_n(b_n) = 1$; (3) $N = \iint_R |\phi_n(z)|^2 dS = \text{minimum}$. The main result is that any circle lying interior to R and enclosing in its interior all the limit points of the b_n contains no zero of any $\phi_n(z)$ other than b_k, $k = 1, 2, \cdots$, if n is taken sufficiently large. The proof uses the normalized functions $\phi_n^*(z) = \phi_n(z)/N$, which, being uniformly bounded on any closed subset Q of R, approach zero uniformly on Q. Since the $|\phi_n(z)|$ are not small only near the boundary of R, the situation is similar to that in which the double integral is replaced by a line integral over the boundary of R. The latter case was treated earlier by the same authors [Trans. Amer. Math. Soc. 79 (1955), 158–172; MR 16, 1011]. *M. Marden.*

Tumarkin, G. C. On integrals of Cauchy-Stieltjes type. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 4(70), 163–166. (Russian)

The author considers integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta} - z} d\sigma(\theta)$$

of Cauchy-Stieltjes type, where $\sigma(\theta)$ is a complex function on $0\leq\theta\leq 2\pi$ with $\int_0^{2\pi}|d\sigma(\theta)|<C$. An integral of this type defines an analytic function $f_1(z)$ in $|z|<1$ and a function $f_2(z)$ which is analytic in $|z|>1$ with $f_2(\infty)=0$. The author considers the converse problem and shows that, given $f_1(z)$ and $f_2(z)$, analytic in $|z|<1$ and $|z|>1$ respectively, with $f_2(\infty)=0$, a necessary and sufficient condition that $f_1(z)$ and $f_2(z)$ be related by such a Cauchy-Stieltjes representation is that there exist a constant $C>0$ such that, for all r in $0<r<1$,

$$\int_0^{2\pi}|f_1(re^{i\theta})-f_2\left(\frac{1}{r}e^{i\theta}\right)|d\theta<C.$$

A further condition is given, under which a function $f(z)$, analytic in $|z|<1$, possesses a Cauchy-Stieltjes integral representation. *A. J. Lohwater* (Ann Arbor, Mich.).

Ul'yanov, P. L. On the A-Cauchy integral. I. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 5(71), 223–229. (Russian)

The A-integral of $\phi(x)$ is the limit of the L-integral of the function obtained by truncating $\phi(x)$ at $\pm n$, where the measure of the set where $|\phi(x)|\geqq n$ is $O(1/n)$. The author shows that if a real f belongs to L on the unit circumference, if u is its Poisson integral, v is a harmonic function conjugate to u, and $F=u+iv$, then $F(z)$ is the A-Cauchy integral

$$(2\pi i)^{-1}(A)\int_{|\zeta|=1}(\zeta-z)^{-1}F(\zeta)d\zeta$$

of its angular boundary values $F(e^{i\theta})$, and the imaginary part of $F(e^{i\theta})$ is the ordinary conjugate integral of f. Further, if an analytic function is represented by a Lebesgue-Cauchy integral, it is represented by the A-Cauchy integral of its boundary values. Various corollaries are obtained. *R. P. Boas, Jr.* (Evanston, Ill.).

Piranian, G.; and Rudin, W. Lusin's theorem on areas of conformal maps. Michigan Math. J. 3 (1955–1956), 191–199.

Lusin [Bull. Calcutta Math. Soc. 20 (1930), 139–154] showed that if $f(z)$ is analytic and of class H_2 in $|z|<1$, there exists a convex domain internally tangent to $|z|=1$ at $z=1$ with the property that the area $A(f, D, \theta)$ of the Riemann surface onto which $w=f(e^{i\theta}z)$ maps D is an integrable function of θ. The authors give two proofs of Lusin's theorem, the second of which relates the domain D and the Taylor coefficients of $f(z)$. It is also shown, among other related results, that if $f(z)$ is meromorphic in $|z|<1$, then, for almost all $e^{i\theta}$ at which the angular cluster set is not total, there exists a convex domain $D(\theta)$, internally tangent to $|z|=1$ at $e^{i\theta}$, such that the image of $D(\theta)$ under $f(z)$ has finite area.

A. J. Lohwater (Ann Arbor, Mich.).

★**Markuschewitsch, A. I. Komplexe Zahlen und konforme Abbildungen.** Kleine Ergänzungsreihe zu den Hochschulbüchern für Mathematik, XVI. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. 56 pp.

Translation, by Gerhard Ränike, of the author's Russian booklet of 1954. Complex numbers and simple functions of complex numbers are discussed, and previous knowledge of complex numbers is not assumed. A strongly geometric viewpoint is taken throughout.

Shibata, Kêichi. Remarks on the sequence of quasiconformal mappings. Proc. Japan Acad. 32 (1956), 665–670.

The author uses the symbol K-QC* for quasiconformal mappings in the Pfluger-Ahlfors sense, and K-QC for quasiconformal mappings of class C'. He proves that every K-QC* mapping can be uniformly approximated by K-QC mappings. The theorem on 2-dimensional absolute continuity is falsely attributed to the reviewer [see A. Mori, Trans. Amer. Math. Soc. 84 (1957), 56–77; MR 18, 646; L. Bers, ibid. 84 (1957), 78–84; MR 18, 646].

L. Ahlfors (Cambridge, Mass.).

Jenkins, James A. A new criterion for quasiconformal mapping. Ann. of Math. (2) 65 (1957), 208–214.

Let $w=f(z)$ be a sense-preserving homeomorphism and set $m(r, z_0)=\min|f(z)-f(z_0)|$, $M(r, z_0)=\max|f(z)-f(z_0)|$ for $|z-z_0|=r$. Further, let $m(z_0)=\lim\inf_{r\to 0}m(r, z_0)/r$, $M(z_0)=\lim\sup_{r\to 0}M(r, z_0)/r$. If $m(z)$ is finite positive and $M(z)\leq Km(z)$ at all points for a finite constant K, it is proved that f is quasiconformal with maximal dilation $\leq K^2$. One would expect that K^2 can be replaced by K.

L. Ahlfors (Cambridge, Mass.).

Piranian, George; and Shields, Allen. The sets of Lusin points of analytic functions. Michigan Math. J. 4 (1957), 15–22.

A point $e^{i\theta}$ on $|z|=1$ is called a Lusin point of a function $f(z)$, analytic in $|z|<1$, if $f(z)$ maps every circle internally tangent to $|z|=1$ at $e^{i\theta}$ onto a domain of infinite area on the Riemann sphere. Lusin [Dokl. Akad. Nauk SSSR (N.S.) 56 (1947), 447–450; MR 9, 181] conjectured that there exists a bounded analytic function for which every point of $|z|=1$ is a Lusin point. This conjecture was proved by Lohwater and Piranian [Michigan Math. J. 3 (1955–56), 63–68; MR 17, 834]. The present authors offer a second proof of the conjecture, and show, in addition, that a necessary and sufficient condition that a set E on $|z|=1$ be the set of Lusin points of some function analytic in $|z|<1$ is that E be of type G_δ. *A. J. Lohwater.*

Baltaga, V. K. On a case of conformal mapping of multiply connected regions. Har'kov. Gos. Univ. Uč. Zap. 29=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 21(1949), 169–183 (1 plate). (Russian)

In the w-plane, let there be given two disjunct closed polygonal domains, degenerate or not, each bounded by a finite number of rectilinear segments. Let P be a complex number with the property that if the w-plane is subjected to a translation through all integral multiples of P, the initial polygons, as well as their images, form a set of pairwise disjunct domains. On removing these domains from the w-plane, an infinitely multiply-connected domain is obtained. The author determines an explicit expression for functions which map conformally the universal covering surface of this domain on the circle $|z|<1$. This formula is analogous to the Schwarz-Christoffel formula and involves elliptic functions. A similar result for the case of one initial polygon was obtained earlier by Achyéser [Soobšč. Har'kov. Mat. Obšč. (4) 9 (1934), 3–8].

W. Seidel (Notre Dame, Ind.).

Grötzsch, Herbert. Zum Häufungsprinzip der analytischen Funktionen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 5 (1956), 1095–1100.

L'A. revient sur ses travaux antérieurs [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 81 (1929), 51–86; 83 (1931), 238–253; 87 (1935), 319–324] pour donner, dans certains cas, un procédé constructif pour la représentation conforme d'un domaine plan de connexion

infinie, sur un domaine à fentes circulaires, sans faire appel aux propriétés des familles normales. Il donne deux lemmes intéressants. *J. Lelong* (Paris).

Tamura, Jirô. A prolongable Riemann surface. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 6 (1956), 123–127.

The author shows by construction of a counterexample that a certain condition stated by de Possel to be necessary for a Riemann surface to be continuable is not valid. {The reviewer wishes to point out that in his own treatment of this question [Ann. of Math. (2) 43 (1942), 280–297, pp. 289, 290; MR 4, 77] an unjustified assumption is made pertaining to the existence of accessible boundary points.} *M. Heins* (Princeton, N.J.).

Ahlfors, Lars V. Square-integrable differentials on open Riemann surfaces. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 758–760.

The author considers various decompositions of the space Γ_h of quadratically integrable harmonic differentials on an open Riemann surface W, which generalize the fact that on a finite Riemann surface every harmonic differential decomposes orthogonally into the sum of a differential of an analytic function, the differential of a harmonic measure, and a harmonic differential whose normal component on the boundary vanishes. The Schottky differentials of W are defined as the limits of the Schottky differentials of finite Riemann surfaces which exhaust W, and it is proved that the orthogonal complement in Γ_h of the space of Schottky differentials of W is the space of harmonic differentials which are both exact and coexact. This implies that all harmonic differentials of W are Schottky if and only if $W \in O_{AD}$.

If the "harmonic measures" of W are defined to be the limits of differentials of the harmonic measures of an exhaustion of W, then they are the orthogonal complement in Γ_h of the space of harmonic differentials whose duals have no periods around dividing cycles of W.

Let Γ_{h0} denote those harmonic differentials which are orthogonal to all coexact differentials. Then to each cycle c on W there is a unique differential $\omega_c \in \Gamma_{h0}$ such that for each closed square-integrable σ

$$\int_c \sigma = \iint_W \sigma \wedge \omega_c.$$

The differentials ω_c span Γ_{h0} and those for which c is a dividing cycle span the harmonic measures.

The author then goes on to discuss conditions under which the generalized Riemann bilinear relation holds for W. *H. L. Royden* (Palo Alto, Calif.).

Ĭrošin, G. D. On the convergence of a subsequence of partial sums of a series to an entire function of finite order. Mat. Sb. N.S. 39(81) (1956), 433–446. (Russian)

I. I. Repin [Mat. Sb. N.S. 36(73) (1955), 3–24; MR 16, 808] discussed the properties of the limit of a sequence $\psi_n(z) = \sum_{j=1}^{n} \gamma_{n,j} f(\lambda_j z)$ when $f(z)$ is an entire function of order ρ and the ψ_n are subjected to a uniform growth condition. Under Repin's hypotheses the $\gamma_{n,j}$ converge to limits γ_j and the series (*) $\sum \gamma_j f(\lambda_j z)$ converges to $\lim \psi_n(z)$. The author weakens one of Repin's hypotheses so that (*) may diverge, and shows that it is still true that a sequence of its partial sums converges. He requires an auxiliary theorem on constructing an entire function of order ρ taking prescribed values at a sequence $\{\lambda_n\}$ whose exponent of convergence is ρ. *R. P. Boas, Jr.*

Leont'ev, A. F. On the region of regularity of the limit function of a certain sequence of analytic functions. Mat. Sb. N.S. 39(81) (1956), 405–422. (Russian)

The author considers the domain of regularity D of the limit of a convergent sequence of entire functions $g_n(z) = \sum_{k=1}^{p_n} a_{nk} f(\lambda_k z)$ (f entire). If $f(z) = e^z$, $\lim_{n\to\infty} n/\lambda_n = 0$, then it is known that D is schlicht and simply connected. An example shows that this conclusion does not hold for every entire function of finite order. It remains true, however, if $f(z) = 1 + \sum_{n=1}^{\infty} z^n / P(1)P(2)\cdots P(n)$, where $P(x)$ is a polynomial with $P(0)=0$, $P(k) \neq 0$ ($k=1, 2, \cdots$). The paper also contains remarks about the position of the singularities of the limit function and about the region of convergence of $\sum \alpha_k f(\lambda_k z)$. *W. H. J. Fuchs.*

Constantinescu, Corneliu. Quelques applications du principe de la métrique hyperbolique. C. R. Acad. Sci. Paris 242 (1956), 3035–3038.

Applying the principle of hyperbolic metric, the author extends a well-known theorem of Landau. Denote by (z_1, z_2) the spherical distance between z_1 and z_2 and let $\varepsilon(a, b, c) = \min\{(a, b), (b, c), (c, a)\}$. Define the dispersion of a closed set F by $\varepsilon(F) = \sup_{a,b,c \in F} \varepsilon(a, b, c)$. The author proves: Let $w(z)$ be a meromorphic function in $|z| < R$. If the set of values omitted by $w(z)$ has a dispersion greater than ε, then $R \leq \operatorname{ctg} \frac{1}{2}\varepsilon / (2B\mathscr{D}f(0))$ where B is an absolute positive constant and $\mathscr{D}f(0)$ is the spherical derivative of $w(z)$ at $z=0$. Furthermore, by using the same principle, the author completes a well-known theorem of Julia.

K. Noshiro (Nagoya).

★ **Priwalow, I. I. Randeigenschaften analytischer Funktionen.** Zweite, unter Redaktion von A. I. Markuschewitsch überarbeitete und ergänzte Auflage. Hochschulbücher für Mathematik, Bd. 25. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. viii+247 pp.

A translation by Joachim Auth, with the editorship of Brigitte Mai, of the Russian 1950 edition [MR 13, 926].

Storvick, David A. On meromorphic functions of bounded characteristic. Proc. Amer. Math. Soc. 8 (1957), 32–38.

Let $w = f(z)$ be meromorphic and of bounded characteristic in $|z| < 1$. Then, $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists and is finite for almost all θ in $0 \leq \theta < 2\pi$. If $|f^*(e^{i\theta})| = 1$ for almost all θ, the author defines $f(z)$ to be of class (B) in $|z| < 1$. Holomorphic functions with the same properties are said to be of class (A). A function of class (B) is said to be non-trivial if neither $f(z)$ nor $[f(z)]^{-1}$ is of class (A). It is shown that if $f(z)$ is a non-trivial function of class (B), then either every point of $|w| = 1$ which is assumed by $f(z)$ at most a finite number of times in $|z| < 1$ is an asymptotic value of $f(z)$, or else $f(z)$ maps $|z| < 1$ in a $(p, 1)$ manner onto a simply connected region consisting of the w-plane slit along an arc of $|w| = 1$.

Again, let $w = f(z)$ be meromorphic in $|z| < 1$ and let the values assumed by $f(z)$ in $|z| < 1$ lie in some domain G of the w-plane whose boundary Γ has positive capacity. Then $f(z)$ is of bounded characteristic in $|z| < 1$. If $f^*(e^{i\theta})$ belongs to Γ for almost all values of θ, $f(z)$ is said to be of class (L) in $|z| < 1$. It is shown, under these conditions, that every arcwise accessible point of Γ is a radial limit value of $f(z)$.

In conclusion, an example is given to show that the class (B) is not contained in the class (L).

[Cf. also in this connection Lehto, Ann. Acad. Sci. Fenn. Ser. A.I. no. **177** (1954); MR **16**, 688; and Noshiro, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 398–401; MR **17**, 143.]

W. Seidel (Notre Dame, Ind.).

Pogorzelski, W. Problème aux limites d'Hilbert généralisé. Ann. Polon. Math. **2** (1955), 136–144.

The results and method of this paper were indicated by the author in a previous publication [Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 367–370; MR **16**, 683]. In the review of the earlier paper it was indicated that a detailed development was to appear later. The present paper gives these details, showing that the conditions of J. Schauder's fixed-point theorem are fulfilled and that the integral equations have the desired solution.

M. S. Robertson (New Brunswick, N.J.).

Lohwater, A. J.; and Piranian, G. The sets of ambiguous points of functions of bounded characteristic. Michigan Math. J. **4** (1957), 23–24.

It is shown that if E is an enumerable subset of $|z|=1$, then there exists a holomorphic function of bounded characteristic in $|z|<1$, whose set of ambiguous points is E. This sharpens a result of the reviewer and Seidel [same J. **3** (1955–56), 77–81; MR **17**, 834].

F. Bagemihl (Notre Dame, Ind.).

Nehari, Zeev. On the coefficients of univalent functions. Proc. Amer. Math. Soc. **8** (1957), 291–293.

Let S be the class of schlicht functions

$$f(z)=z+a_2z^2+a_3z^3+\cdots$$

in $|z|<1$, $A_n=\sup|a_n|$ for f in S, and $\alpha=\limsup A_n/n$. It is shown that if $f(z)\neq d$ in $|z|<1$, then $|a_n|\leq 4|d|\alpha n$. The proof is based on the fact that if $F(z)=b_1z+b_2z^2+\cdots$ is schlicht in $|z|<1$ and omits the value 1 there, then $2F(z^2)-2\{F(z^2)[F(z^2)-1]\}^{\frac{1}{2}}$ is also schlicht there.

P. R. Garabedian (Stanford, Calif.).

Šaginyan, A. L. On the theory of univalent functions. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **9** (1956), no. 7, 29–35. (Russian. Armenian summary)

Soit E un ensemble fermé borné dont le complémentaire est simplement connexe; $w=\varphi(z)$ la fonction représentant conformément CE sur $|w|>1$, $\varphi(\infty)=\infty$, $\varphi'(\infty)=\tau^{-1}>0$; L une courbe rectifiable entourant E; d (resp. $d+d_0$) la plus grande distance à l'origine des points de E (resp. L). L'auteur montre que: I) pour tout $z\in L$ on a

$$\log|\varphi(z)|>\log\frac{d+d_0}{d}\cdot\exp\Big(-\int_L\frac{ds}{\rho(s)}\Big)=A,$$

où $\rho(s)$ est la distance à E du point d'abcisse s sur L; II) pour tout couple $z_1\in L$, $z_2\in L$

$$|\varphi(z_2)-\varphi(z_1)|\geq A(4\tau+2d_0)^{-1}l,$$

où l est la borne inférieure des longueurs des arcs joignant z_1 et z_2 dans CE. *J. P. Kahane* (Montpellier).

Lavrent'ev, M. M. Quantitative estimates of interior theorems of uniqueness. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 731–734. (Russian)

Let $f(z)$ be analytic in D: $|z|<1$ and bounded, $|f(z)|<1$, and let $|f(z)|\leq\varepsilon$ on a set $A\subset D$; it is assumed that A contains a sequence $\{a_k\}$, $|a_k|\leq|a_{k+1}|$ with a point a of D as a limit point. If $m(r)$ denotes the minimum modulus of $f(z)$ on $|z|=r<1$, then, for arbitrary k, either 1) there exists an r_k', $|a_k|\leq r_k'\leq|a_{k+2}|$, such that $m(r_k')=0$; or 2) $m(r)\leq\varepsilon$ on one of two segments $|a_k|\leq r\leq|a_{k+1}|$, $|a_{k+1}|\leq r\leq|a_{k+2}|$. This result is then used to obtain various bounds of $|f(z_0)|$ for z_0 in D. *A. J. Lohwater.*

Stečkin, S. B. An extremal problem for polynomials. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 765–774. (Russian)

The problem is to determine

$$M_n^{(p)}(D)=\sup\sum_{k=0}^{n}d_k^{2-p}|c_k|^p,$$

where $D=\{d_n\}$ is a given sequence of non-negative numbers, $0<p<2$, and the supremum is taken over all polynomials $p_n(z)=\sum_{k=0}^n c_kz^k$ such that $|p_n(z)|\leq 1$ for $|z|\leq 1$. The author shows that the ratio of $M_n^{(p)}(D)$ to $\{\sum_{k=0}^n d_k^2\}^{1-\frac{1}{2}p}$ is bounded above and below by positive absolute constants. As an application he constructs, for any D with $\sum d_n^2=\infty$, a power series $f(z)$ which is continuous (but not uniformly convergent) in the closed unit disk and has $\sum d_n^{2-p}|c_n|^p=\infty$. *R. P. Boas, Jr.*

Havinson, S. Ya. Extremal problems for certain classes of analytic functions in finitely connected regions. Amer. Math. Soc. Transl. (2) **5** (1957), 1–33.

A translation of the 1955 Russian paper reviewed in MR **17**, 247.

Leont'ev, A. F. On properties of sequences of linear aggregates that converge in a region in which the system of functions generating the linear aggregates is not complete. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 26–37. (Russian)

The situation described in the title is that in which a set $\{f_n(z)\}$ of analytic functions is not complete in a region and one investigates the properties of the linear manifold spanned by $\{f_n(z)\}$, i.e. the properties of limits of sums $\sum_{k=1}^n a_{kn}f_k(z)$. The author reviews, without proofs, the literature of this (fairly young) field. *R. P. Boas, Jr.*

Andreoli, Giulio. Sulla polidromia di funzioni assegnate con espressioni aritmetiche e sul teorema di Volterra-Poincaré. Ricerca, Napoli **7** (1956), 32–42.

Diskussion der Menge aller Funktionen von der Form $\sum_{n=1}^\infty A_n\sqrt{(z-a_n)}$ wobei die Folgen A_n und a_n so gewählt werden, dass die Reihe für alle z konvergiert. Die Vorzeichen der Wurzeln können beliebig gewählt werden.

W. Saxer (Zürich).

Al'per, S. Ya. On the convergence of Lagrange's interpolational polynomials in the complex domain. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 44–50. (Russian)

Let D be a bounded domain of the z-plane whose boundary is a simple closed curve C. Under a smoothness condition on C it is possible to choose points $\{z_{mn}\}$ ($m=1, 2, \cdots$; $n\leq m$) such that the Lagrange interpolation polynomials $L_m(z;f)$ defined by $\deg L_m\leq m-1$, $L_m(z_{mn})=f(z_{mn})$ ($n=1, 2, \cdots, m$) converge uniformly to $f(z)$ in $\bar{D}$ whenever $f(z)$ is regular in D and continuous in $\bar{D}$ with a modulus of continuity $\omega(\delta)=o(1/(-\log\delta))$.

The theorem ceases to be true if no assumption is made about the modulus of continuity. *W. H. J. Fuchs.*

Džrbašyan, M. M.; and Hačatryan, I. O. On the completeness of the system of functions $\{z^{\lambda_n}\}$ in the complex domain for weighted square approximation. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 914–917. (Russian)

Let $B(\alpha)$ be the Hilbert space of functions $f(z)$ regular in $|\arg z|<\pi/2\alpha$ $(\frac{1}{2}\leqq\alpha<\infty)$ with norm

$$\|f\|^2=\int_{-\pi/2\alpha}^{\pi/2\alpha}d\theta\int_0^\infty e^{-P(r)}|f(re^{i\theta})|^2 r\,dr,$$

where

$$P(r)=\text{const}+\int_1^r\frac{\omega(t)}{t}dt,\ \omega(t)\uparrow\infty \text{ as } t\to\infty.$$

Theorem. If $\{\lambda_n\}$ is a sequence of positive numbers, $\lambda_{n+1}-\lambda_n\geqq h>0$, $\lambda(r)=2\sum_{\lambda_n<r}\lambda_n^{-1}$, then $\{z^{\lambda_n}\}$ is complete in $B(\alpha)$, if for every b, $0<b\leqq1$, $\int_1^\infty P(b\exp(k(r)))r^{-2}dr=\infty$, where $k(r)=\inf_{s\geqq r}\{\lambda(s)-(1/\alpha)\log s\}$.

A similar theorem is proved for mean-square approximation with weight function $\exp(-P(r))$ on the angle $L(\alpha)$ defined by $\arg z=\pm\pi/2\alpha$. *W. H. J. Fuchs.*

Ibragimov, I. I. On mean square approximation of functions of a complex variable in infinite regions by entire functions of finite degree. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 50–56. (Russian)

The author considers functions defined on an unbounded simply-connected region D and belonging to $L^p(D)$ in a natural sense. He defines the best mean-p approximation $A_\nu^{(p)}(f;D)$ to f by entire functions of exponential type ν in the natural way (using double integrals), as well as an integral pth power modulus of continuity $\omega_p(f;D)$ over D. For $L^2(D)$ he characterizes the entire function of exponential type ν of best approximation to a given f. He shows that

$$\omega_p(f,\delta)\leqq 2A_\nu^{(p)}(f;D)+\nu\delta M,$$

where M is independent of ν and δ, and that

$$A_\nu^{(p)}(f;D)\leqq C\omega_p(f;1/\nu).$$

R. P. Boas, Jr. (Evanston, Ill.).

Leray, Jean. Fonction de variables complexes: sa représentation comme somme de puissances négatives de fonctions linéaires. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 589–590.

Formule simple qui permet de représenter une fonction analytique de l variables par une somme de puissances $(-l)$ièmes de fonctions linéaires de ces variables et qui généralise la formule classique de Cauchy, relative à $l=1$. C'est un cas particulier de résultats de l'auteur relatifs au produit fonctionnel projectif de L. Fantappiè [cf. J. Leray, C. R. Acad. Sci. Paris **242** (1956), 953–959; MR **17**, 1093]. Pas de démonstration. *H. G. Garnir* (Liège).

Fuks, B. A. Some new results in the theory of analytic functions of several variables. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 85–93. (Russian)

The author gives a brief account of well-known results by H. Cartan, S. Bergmann, P. Thullen, A. Weil, K. Oka, K. Stein, J. P. Serre, and others on analytic manifolds, regularity domains, and generalizations of Cauchy's formula to analytic functions of several variables. No proofs are given and no fresh results. *H. Tornehave.*

See also: Robinson, p. 713; Gheorghiu, p. 730; Voronovskaya, p. 730; Ostrowski, p. 732; Kimura, p. 738; Matthies, p. 739; Volkovyskiĭ, p. 740; Belinskiĭ, p. 740; Berg, p. 741; Bergman, p. 741; Bicadze, p. 743; Ehrenpreis, p. 746; Arcidiacono, p. 756; Vesentini, p. 763.

Geometrical Analysis

See: De Sloovere, p. 744; Švec, p. 761; Vesentini, p. 763; Bodiou, p. 779.

Functions with Particular Properties

Naïm, Linda. Sur l'allure à la frontière des fonctions harmoniques positives. C. R. Acad. Sci. Paris **243** (1956), 1266–1268.

Poursuivant ses études antérieures [voir surtout mêmes C. R. **241** (1955), 1907–1910, pp. 1907–1908; MR **17**, 1073] l'auteur étudie les fonctions harmoniques >0 dans un espace de Green. On donne d'abord avec la frontière de Martin des résultats concernant l'allure du quotient u/h (la fonction harmonique >0 fixe, un solution $\mathfrak{D}_{f,h}$ ou fonction harmonique >0 quelconque) au voisinage d'un point minimal où G_{p_0}/h ne tend pas vers 0; cela étend l'étude connue des fonctions harmoniques au voisinage d'un point-frontière irrégulier d'un domaine euclidien. D'autre part, avec une frontière métrisable compacte quelconque assurant seulement la h-résolutivité des données finies continues, on approfondit une notion de filtre $\mathcal{F}$ convergeant vers un point-frontière Q et h-indépendant, c'est à dire tel que, pour toute donnée f qui soit h-résolutive, $\mathfrak{D}_{f,h}/h$ admette une lim sup$_{\mathcal{F}}$ majorée par la lim sup de f en Q, en supposant f bornée supérieurement seulement au voisinage de Q. Cela tourne les difficultés de „l'action à distance" de f qui peut donner une lim sup ordinaire infinie en Q lorsque f n'est pas bornée supérieurement. *M. Brelot* (Paris).

Vidav, Ivan. Sur une généralisation du théorème de Mandelbrojt-MacLane aux fonctions harmoniques et sousharmoniques. Bull. Soc. Math. Phys. Serbie **6** (1954), 123–130.

The present paper contains proofs of results previously announced by the author [C. R. Acad. Sci. Paris **238** (1954), 2483–2485; MR **15**, 956] concerning harmonic and subharmonic functions and generalizing a theorem of Mandelbrojt and MacLane [ibid. **223** (1946), 186–188; MR **8**, 20] relative to the identical vanishing of a holomorphic function defined in a strip. For example, let Δ_s be a domain in the complex $s=\sigma+it$ plane, defined by $\sigma>\sigma_0$, $-G_1(\sigma)<t<G_2(\sigma)$, where $G_1(\sigma)$ and $G_2(\sigma)$ are positive continuous functions of bounded variation such that $G_1(\sigma)\to G_1>0$, $G_2(\sigma)\to0$ as $\sigma\to\infty$; let $w(s)$ be subharmonic and bounded from above in Δ_s; let $N(\sigma)$ be a nondecreasing function such that $\int^\infty N(\sigma)e^{-S(\sigma)}d\sigma=\infty$, where $S(\sigma)=\pi\int_{\sigma_0}^{\sigma}[G_1(t)+G_2(t)]^{-1}dt$. It is shown that if $w(s)$ satisfies $w(\sigma+it)<-N(\sigma)$ then $w(s)$ must satisfy $w(s)\equiv-\infty$. *E. F. Beckenbach.*

Weinstein, Alexandre. Sur un problème de Cauchy avec des données sousharmoniques. C. R. Acad. Sci. Paris **243** (1956), 1993–1994.

Soit x un point variable $(x_1,\cdots,x_m)$ de R^m. Appelons $u^{(k)}(x,t,f)$ la solution unique de problème de Cauchy pour l'équation

$$\frac{\partial}{\partial t}\left(t^k\frac{\partial u}{\partial t}\right)=t^k\sum\frac{\partial^2u}{\partial x_i^2}\ (k\geqq0)$$

avec la donnée $u=f(x)$, $\partial u/\partial t=0$ pour $t=0$. Grâce à une relation entre $u^{(\alpha)}$ et $u^{(\beta)}$, on obtient une série de propriétés de maximum ou de convexité. Par exemple, supposons

$k \geq m-1$; alors le maximum de $u^{(k)}$ vaut celui de f et si f est sousharmonique, $u^{(k)}$ est décroisante de k et convexe de $t^{2-m}(m>2)$ ou log t $(m=2)$. *M. Brelot* (Paris).

Ghermănescu, M. Sur les fonctions n-harmoniques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 529–536. (Romanian. Russian and French summaries)

In a previous paper [Bull. Sci. Ecole Polytech. Timişoara **2** (1929), 188–194] the author has defined the concept of an (n)-holomorphic function. $f(z)$ is said to be (n)-holomorphic if and only if its nth areolaric derivative vanishes, or, equivalently, if $f(z)$ is a polynomial of degree $n-1$ in $\bar{z}$, whose coefficients are analytic functions of z. The author now shows that a function $u(x, y)$ is n-harmonic, i.e. satisfies the equation $\Delta^n u=0$, if and only if it is the real part of an (n)-holomorphic function. As an application, Riquier's problem (the determination of $u(x, y)$, satisfying $\Delta^n u=0$ in a simply connected domain, with preassigned values for $\Delta^k u$ $(k=0, 1, \cdots, n-1)$ on the boundary) is solved, by reducing it to the successive solution of n Dirichlet problems. In the particular case $n=1$ the results are classical and for $n=2$, they were given by Goursat. *E. Grosswald* (Philadelphia, Pa.).

Duff, G. F. D. On the Neumann and dual-adjoint problems of generalized potential theory. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 23–31.

Partant de la solution du problème mixte ($t\phi$ et $t\delta\phi$ donnés sur la frontière) résolu par lui [Canad. J. Math. **6** (1954), 427–440; MR **17**, 474] pour l'équation $\Delta\varphi=0$, l'A. donne ici une nouvelle démonstration de l'existence de solutions du problème de Neumann ($t\delta\varphi$ et $nd\varphi$ donnés sur la frontière) déjà établie par lui [ibid. **5** (1953), 196–210; MR **14**, 903]. φ désigne toujours une forme différentielle sur une variété riemanienne à bord régulier La même méthode lui permet d'établir l'existence de solutions pour le problème nouveau, "dual-adjoint" du précédent, obtenu en donnant $t\varphi$ et $t{*}d\varphi$ sur la frontière. Les données sont assujetties à vérifier une infinité de conditions intégrales, entraînant des conditions ponctuelles. *J. Lelong* (Paris).

Górski, J. Sur certaines propriétés de points extrémaux liés à un domaine plan. Ann. Polon. Math. **3** (1956), 32–36.

Dans le plan soit F un compact-frontière formé de m composantes connexes non ponctuelles F_i et f finie continue réelle sur F. On introduit la ω-distance de deux points z, ζ sur F par $|z-\zeta| \exp\{-\lambda[f(z)+f(\zeta)]\}$ ($\lambda>0$ fixé). Un système de $n+1$ points de F est dit extrémal s'il réalise le maximum du produit des ω-distances des divers couples du système. On considère des systèmes extrémaux pour $n=1, 2, \cdots$ et on désigne par $F_{\lambda f}$ l'ensemble des points d'accumulation de l'ensemble total. On démontre grâce aux travaux récents de Leja [mêmes Ann. **1** (1954), 13–28; MR **16**, 348] que, lorsque f est nul sur F_1 et constant et assez petit sur les autres F_i, alors $F_f=F$. *M. Brelot* (Paris).

Szybiak, A. Some properties of plane sets with positive transfinite diameter. Ann. Polon. Math. **3** (1956), 19–28.

Reprenant dans le plan le diamètre transfini et un polynôme d'approximation qui conduit selon Leja à la fonction de Green, l'auteur donne de nouvelles démonstrations de résultats anciens comme le lemme de Kellog (existence pour un compact non de capacité nulle, d'un point-frontière régulier pour le domaine infini complémentaire). *M. Brelot* (Paris).

See also: Ahlfors, p. 727; Pini, p. 739; Bergman, p. 741; Mergelyan, p. 734; Ehrenpreis, p. 746.

Special Functions

Bhonsle, B. R. On a series of Rainville involving Legendre polynomials. Proc. Amer. Math. Soc. **8** (1957), 10–14.

The author derives expressions for various integrals involving the Legendre polynomials $P_n(x)$ in the form of finite series, and the results are often stated in terms of the hypergeometric functions ${}_iF_j$. The investigation is carried out with the aid of E. D. Rainville's series for $P_n(x)$ [Bull. Amer. Math. Soc. **51** (1945), 268–271; MR **6**, 211]. *C. A. Swanson* (Pasadena, Calif.).

Arscott, F. M. On Lamé polynomials. J. London Math. Soc. **32** (1957), 37–48.

Lamé polynomials are solutions of the differential equation $d^2w/du^2+[h-n(n+1)k^2 \operatorname{sn}^2 u]w=0$. A new notation is proposed and new expansions are given in terms of solutions of Gegenbauer's equation. They contain a parameter and reduce in special cases to expansions in terms of Legendre polynomials or Tchebycheff polynomials. *J. Meixner* (Aachen).

Voronovskaya, E. V. The method of functionals applied to polynomials of N. I. Ahiezer. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 727–730. (Russian)

The equations $F(x^j)=\mu_j$ $(j = 0, 1, 2, \cdots, n)$ define a linear functional F on the polynomials of degree $\leq n$. Continuing previous work [same Dokl. (N.S.) **99** (1954), 5–8, 193–196; MR **17**, 842] the author investigates polynomials $Q(x)$ maximising a functional F subject to $\sup_{0\leq x\leq 1} |Q(x)|=1$. The Q considered in this paper are of degree n and attain absolute value 1 at $x_1=0$, x_2, $\cdots$, $x_{n-1}=1$ $(x_1<x_2\cdots)$ while $Q(x_j)Q(x_{j+1})=-1$. It is stated that these Q arise from extremal problems with $\mu_j=0$ $(j\leq n-3)$, $\mu_{n-2}=1$, $\mu_{n-1}=u$, $\mu_n=v$. Partial differential equations for the coefficients of Q as functions of u and v are given. If $P(x)=\sum p_j x^j$ is a polynomial of degree n maximising $\sup_{-1\leq x\leq 1} |P(x)|$ subject to $p_n=1$, $p_{n-1}=\lambda$, $p_{n-2}=\mu$, then $P(x)$ is of the form $cQ(2x-1)$, where Q is in the class of polynomials described above. *W. H. J. Fuchs* (Ithaca, N.Y.).

Carlitz, L. A formula for the product of two Hermite polynomials. J. London Math. Soc. **32** (1957), 94–97.

Let

$$H^*\varphi= \frac{2^{\frac{1}{2}(m+n)}m!n!}{(m+n)!}H_{m+n}\{x(1+\sec\varphi)^{\frac{1}{2}}\}\cos^{\frac{1}{2}(m+n)}\varphi \cos\tfrac{1}{2}(m-n)\varphi.$$

Bailey [same J. **13** (1938), 202–203] proved that

$$H_m(x)H_n(x)=\frac{1}{\pi}\int_0^\pi H^*(\varphi)d\varphi.$$

Without using this formula Carlitz now proves the discrete analogue

$$H_m(x)H_n(x)=\frac{1}{t}\sum_{v=0}^{t-1} H^*\left(\frac{2\pi v}{t}\right) \quad (t>\max(m, n))$$

(together with two deductions), saying that this result was suggested by the reviewer's formula

$$P_n(x)=\frac{1}{t}\sum_{v=0}^{t-1}\left\{x+(x^2-1)^{\frac{1}{2}}\cos\frac{2\pi v}{t}\right\}^n$$

[Proc. Cambridge Philos. Soc. **51** (1955), 385–388; MR **16**, 1019]. {Carlitz's formula could also be deduced from Bailey's by showing that $H^*(\varphi)=\sum a_\nu e^{i\nu\varphi}$ where $a_\nu=0$ if $|\nu|>\max(m, n)$ and then applying formula (2) of the reviewer's paper.} *I. J. Good* (Cheltenham).

★**Satake, Ichiro. On Siegel's modular functions.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 107–129. Science Council of Japan, Tokyo, 1956.

Let $\mathfrak{H}_n$ be the space of all complex symmetric matrices $Z=X+iY$ of order n, with $Y>0$ and let M_n be Siegel's modular group of degree n; then Siegel's modular functions $F(Z)$ are defined as quotients of two modular forms on $\mathfrak{V}_n=M_n\backslash\mathfrak{H}_n$, regular at the points at infinity. If one wants to define them directly as meromorphic functions on $\mathfrak{V}_n$, difficulties arise because the quotient space $M_n\backslash\mathfrak{H}_n$ is not compact. The author tries to obviate these difficulties by a compactification procedure. In order to do that, he introduces complex analytic manifolds with ramifications, called V-manifolds. Their precise definition is rather complicated and cannot be recorded here, but it is possible to define meromorphic functions on a V-manifold and the $\mathfrak{V}_n$, in particular, are V-manifolds. Denote by $\mathfrak{W}_{n-1}$ the V-manifold consisting of the equivalence classes under M_n of the limit points of the sequences $Z^{(k)}=(z_{ij}^{(k)})$, such that $\lim_{k\to\infty} y_{nn}^{(k)}\to\infty$, while all other elements stay bounded. Next, a "junction" of V-manifolds $\mathfrak{V}_n$ and $\mathfrak{W}_m$ ($m<n$, $\mathfrak{V}_n\cap\mathfrak{W}_m=\emptyset$) is defined as the V-manifold sturcture on $\mathfrak{V}_n\cup\mathfrak{W}_m$, such that $\mathfrak{V}_n$ becomes an open V-submanifold and $\mathfrak{W}_m$ a regularly imbedded V-submanifold of $\mathfrak{V}_n\cup\mathfrak{W}_m$, in case such a V-structure can be defined. In general, taking as $\mathfrak{W}_m$ the previously defined $\mathfrak{W}_{n-1}$, the new V-manifold $\mathfrak{V}_n\cup\mathfrak{W}_{n-1}$ is still not compact. However, if $\mathfrak{W}_{n-1}$ itself is first completed to a compact V-manifold $\overline{\mathfrak{W}}_{n-1}$, and then joined to $\mathfrak{V}_n$, the resulting V-manifold

$$\overline{\mathfrak{V}}_n=\mathfrak{V}_n\cup\overline{\mathfrak{W}}_{n-1}$$

is compact. The author conjectures that this should be possible for every n, as would follow from a proof that $\overline{\mathfrak{V}}_n$ is a projective variety. In the particular case $n=2$, the V-structure of $\mathfrak{W}_1$ is trivial ($\mathfrak{W}_1\approx C\times C$, whence $\overline{\mathfrak{W}}_1\approx\bar{C}\times\bar{C}$, C being the complex plane and $\bar{C}=C\cup\{\infty\}$ the complex sphere) and the author shows that

$$\overline{\mathfrak{V}}_2=\mathfrak{V}_2\cup\overline{\mathfrak{W}}_1$$

is compact. This proof, however, is far from trivial and succeeds mainly because in this case $n-1=1$ and one has at his disposal the well developed theories of theta-functions and of elliptic functions. *E. Grosswald.*

Ma, Min-Yuan. Sur les symboles de Hankel et le calcul de $J_\nu(z)Y_{\nu+p}(z)-Y_\nu(z)J_{\nu+p}(z)$. C. R. Acad. Sci. Paris **243** (1956), 1995–1997.

The author notes some identities satisfied by Hankel's symbols, (n, m), and applies them to generalizing the Wronskian of the Bessel functions, getting an expansion for the expression in the title.

E. Pinney.

Bose, B. N. On certain integrals involving Legendre and ultraspherical polynomials. Ganita **6** (1955), 27–37 (1956).

An integral involving an ultraspherical polynomial and a generalized hypergeometric function is integrated using a Rodrigues' formula and integration by parts and then summing a series. A large number of special cases are noted. Some of the integrals enable the author to make certain expansions, such as expressing $P_{2n}(z)$ in a series of Legendre functions $P_r(1-2z^2)$. *E. Pinney.*

Jaswon, M. A. Limiting properties of Mathieu functions. Proc. Cambridge Philos. Soc. **53** (1957), 111–114.

The differential equation

$$(1)\qquad r\frac{d}{dr}\left(r\frac{d}{dr}\right)y-(a+\alpha^2r^2+\beta^2r^{-2})y=0$$

with parameters a, α, β can be reduced to Mathieu's differential equation. For a equal to a characteristic number a_{2n}, the behavior of one solution of (1) for $\alpha\to0$ or $\beta\to0$ is deduced from properties of the Mathieu function $\mathrm{ce}_{2n}(z; q)$. The results could have been easily obtained by applying formula (41), p. 201 of J. Meixner and F. W. Schaefke, Mathieusche Funktionen und Sphäroidfunktionen [Springer, Berlin, 1954, MR **16**, 586], with $s=\pm n$, $\mathfrak{z}=J$. *J. Meixner* (Aachen).

[illegible]ivastava, H. M. On certain relations involving the generalised K-function of Bateman. Ganita **5** (1954), 183–189 (1955).

Starting from the definition of the functions

$$P_{2n,2k}(x)=(2/\pi)\int_0^{\pi/2}\cos^{2k}\theta\cos(x\tan\theta-2n\theta)d\theta$$

with $n-k=1, 2, 3, \cdots$, which are simply related to the Whittaker functions $W_{n,k+\frac{1}{2}}(2x)$, integrals of the type

$$\int_0^{\pi/2}\cos^{2k}\theta\, f(x, \theta)d\theta,$$

with $f=\cos(x\tan\theta-y\sin\theta)$, $\cos(x\tan\theta)J_0(2z\sin\theta)$, $\cos(x\tan\theta)[p+\sin^2\theta]^{-\frac{1}{2}}$, $\cos(x\tan\theta)\exp(t\cos^2\theta)$ are expanded in terms of the functions $P_{2n,2k}$. *J. Meixner.*

Chaundy, T. W. An integral for Appell's hypergeometric function $F^{(4)}$. Ganita **5** (1954), 231–235 (1955).

A definite integral representation of Appell's function $F_4(a, b; c, c'; x, y)$ has been given by J. L. Burchnall and T. W. Chaundy [Quart. J. Math. Oxford Ser. (2), **11** (1940), 249–270; MR **2**, 287]. It is slightly transformed to give a simplified integrand, while the region of integration loses its simplicity. *J. Meixner* (Aachen).

See also: Zmorovič, p. 713; Manara, p. 713; Baltaga, p. 726; Ghircoiaşiu, p. 731; Sharma and Srivastava, p. 736.

Sequences, Series, Summability

Ghircoiaşiu, Nicolae. Sur le développement en série entière de l'inverse d'un polynôme. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. Ser. I. **6** (1955), 51–77. (Romanian. Russian and French summaries)

If $P(x)$ is a polynomial such that $P(0)=1$, all zeros of $P(x)$ are real, and the zero of least absolute value is positive, then the author proves that in the power series expansion of $1/P(x)$ there are at most a finite number of

negative coefficients. If $P(x)$ is of degree 2, 3, or 4, and the coefficient of x in the said expansion is positive, then all coefficients are positive. There are other similar results, the author discusses also polynomials with real coefficients whose zeros are complex, and he discusses also the zeros of the partial sums of the power series expansion of $1/P(x)$. *A. Erdélyi* (Jerusalem).

Ostrowski, Alexandre. Le développement de Taylor de la fonction inverse. C. R. Acad. Sci. Paris **244** (1957), 429–430.

Let $w=f(z)=\sum_{n=1}^{\infty} a_n z^n$ converge in the neighborhood of the origin and suppose $a_1\neq 0$. Let $z=g(w)=\sum b_n w^n$ be the power series for the inverse function. The author obtains a more workable formula for the coefficients b_n in terms of the coefficients a_n than the classical formula of Lagrange. This new formula is based on a recurrence relation due to Schroeder [Math. Ann. 2 (1870), 317–363, p. 330]. The following expression is deduced for the b_n

$$f_n=\sum (-1)^{n-k_1-1}\frac{(2n-k_1-2)!}{m!k_2!k_3!\cdots k^n!}a_1^{-(2n-k_1-1)}a_2^{k_2}\cdots a_n^{k_n}$$

where the summation is over integers k_n satisfying the conditions $k_n\geq 0$, $\sum_{r=1}^{m} k_r=n-1$ and $\sum_{r=1}^{n} rk_r=2m-2$. *V. F. Cowling* (Lexington, Ky.).

Polniakowski, Z. On certain theorems of the Mercer type. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 24[illegible]-246.

The transformation (H,μ): $t_m=\sum a_{mn}s_n$, where $a_{mn}=\binom{m}{n}\Delta^{m-n}\mu_n$ $(n\leq m)$, $=0$ $(n>m)$ is called the Hausdorff transform of the sequence $\{s_n\}$ into the sequence $\{t_n\}$ relative to the sequence $\{\mu_n\}$. In order that (H,μ) be regular (that is, transform convergent sequences into convergent sequences having the same limit) it is necessary and sufficient that μ_n be a regular moment constant. The author of the present paper states (proofs to be published later) several theorems regarding Hausdorff summation of which the following are examples. If $\mu_n=W_1(n)/W(n)$, where $W(x)$ is a polynomial of degree k, $W_1(x)$ is of degree $l\leq k$, $W(n)\neq 0$ and $W(0)=W_1(0)$ then necessary and sufficient conditions for the regularity of Hausdorff's method corresponding to the sequence $\{\mu_n\}$ is that the real parts of the roots of the equation $W(x)=0$ be negative. This theorem is applied to prove the Mercerian theorem: Let $p_n=\alpha s_n+(1-\alpha)t_n$, $\alpha\neq 0$, $W(n)\neq 0$ for $n=0, 1, \cdots$ and let the sequence $\mu_n=1/w(n)$ correspond to the transformation of the sequence $\{s_n\}$ into the sequence $\{t_n\}$; the assumption $\lim p_n=s$ $(|s|<\infty)$ implies $\lim s_n=s$ if and only if the real parts of all roots of the equation $W(x)=1-1/\alpha$ are negative. *V. F. Cowling* (Lexington, Ky.).

Polniakowski, Z. On some Tauberian theorems. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 651–653.

In this paper the author states without proof generalizations of results of the paper reviewed above concerning Hausdorff transforms and Euler difference equations. A typical result follows. Let $w(z)$ be a polynomial of degree $k\geq 1$ and $w(n)\neq 0$ for $n=0, 1, 2, \cdots$. Suppose 1) $\operatorname{Re}(\alpha)>\max \operatorname{Re}(\rho_\nu)$, where ρ_ν are zeros of $w(z)$, 2) $\limsup |n^{l-\alpha}\Delta^l s_n|\leq M$ for a positive integer l, and 3) $t_n\sim sn^\alpha$, where $\{t_n\}$ is the Hausdorff transform of the sequence $\{s_n\}$ corresponding to the generating sequence $\mu_n=1/w(n)$. Then $s_n\sim sw(\alpha)n^\alpha$. The proof of this theorem depends upon a study of the asymptotic relation satisfied by the solutions, x_n, of the difference equation

$$L(x_n)=L^*(s_n),$$

where

$$L(x_n)=\sum_{\nu=0}^{k}\tau_\nu\binom{n}{\nu}\Delta^\nu x_{n-\nu},\quad L^*(x_n)=\sum_{\nu=0}^{k}\eta_\nu\binom{n}{\nu}\Delta^\nu x_{n-\nu},$$

$$\tau_\nu=(-1)^\nu\Delta^\nu w(0),\quad \eta_\nu=(-1)^\nu\Delta^\nu w_1(0),\quad w(0)=w_1(0).$$

V. F. Cowling (Lexington, Ky.).

Agnew, R. P. Borel transforms of Tauberian series. Math. Z. 67 (1957), 51–62.

Let S_n denote the partial sum $\sum_{k=0}^{n} u_k$ and $B(t)$ the Borel transform $e^{-t}\sum_{k=0}^{\infty} t^k S_k/k!$ of a series of complex u_n satisfying the Tauberian condition

$$(*)\qquad \limsup |n^{\frac12}u_n|=L<\infty.$$

Let $n(\alpha)$ and $t(\alpha)\to\infty$ as $\alpha\to\infty$. The author determines the least constant A such that

$$\limsup_{\alpha\to\infty} |B(t)-S_n|\leq AL$$

for every series $\sum u_n$ satisfying (*). He shows that

$$A=(2\pi)^{-\frac12}\int_{-\infty}^{\infty} e^{-\frac12 x^2}|x-M|dx,$$

where $M=\limsup |n-t|/t^{\frac12}$. In particular A assumes its minimum value $(2/\pi)^{\frac12}$ if and only if $M=0$. In this connection the author notes that Rajagopal [Math. Z. 57 (1953), 405–414; MR 14, 958] proved that $A=(2/\pi)^{\frac12}$ in the special case $n=[t]$. The author shows that his results do not change if (*) is replaced by an appropriate Tauberian condition of Schmidt type. The paper contains useful estimates of the expressions $e^{-t}\sum_k |k^\alpha-n^\alpha|t^k/k!$ for $\alpha=\frac12, 1$ when $n=t+\omega t^{\frac12}\to\infty$, ω bounded. *J. Korevaar*.

Ramanujan, M. S. A note on the quasi-Hausdorff series-to-series transformations. J. London Math. Soc. 32 (1957), 27–32.

A Toeplitz transformation of sequences or series is a quasi-Hausdorff transformation if its matrix is the transpose of a Hausdorff matrix. The author proves some inclusion theorems for such transformations, and by specialization he obtains the following result: If the partial sums of $\sum a_n$ form a bounded sequence and if, for some positive constant α, the series $\sum b_n$ with

$$b_n=\alpha a_n+(1-\alpha)\left[\frac{a_n}{n+1}+\frac{a_{n+1}}{n+2}+\cdots\right]$$

converges, then $\sum a_n$ converges. *G. Piranian*.

Kuttner, B. On cores of sequences and of their transforms by regular matrices. Proc. London Math. Soc. (3) 6 (1956), 561–580.

The author characterizes regular matrices $A=(a_{nk})$ of complex elements such that each complex sequence $s=\{s_0, s_1, \cdots\}$ has a core $C(s)$ containing the core $C(As)$. Such matrices are called core regular. The core $C(s)$ is the intersection of the sets $C_0, C_1, \cdots$ where C_n is the least closed convex set in the (finite) complex plane containing $s_n, s_{n+1}, \cdots$. The core $C(s)$, which contains a single point z_0 if and only if s converges to z_0, may be otherwise described as the set of points z such that

$$\Re\{ze^{i\phi}\}\leq \limsup_{n\to\infty} \Re\{s_n e^{i\phi}\}$$

for each real ϕ. The core $C(As)$ is defined to be the set of

points z such that

$$\Re\{ze^{i\phi}\} \leq \limsup_{n\to\infty} \limsup_{m\to\infty} \Re\{[\sum_{k=0}^{m} a_{nk}s_k]e^{i\phi}\}$$

for each real ϕ. This set $C(As)$ (which, like $C(s)$, may be empty) is called the core of the A transform of s even when A and s are such that some or all of the series in

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk}s_k$$

are divergent and it may be said that σ and As do not exist. It is first shown that A is core regular if and only if $A=B+C$ where B is real and core regular and C is pure imaginary and such that $\lim_{n\to\infty} c_{nk}=0$ for each k and, for some constant q, $c_{nk}=0$ when n, $k>q$. The problem of characterizing core regular matrices is then reduced to nontrivial problems involving transforms of real nonnegative sequences by real matrices. The conditions characterizing core regular matrices are too numerous and complicated for presentation here. The results complement and complete earlier investigations to which the author gives references. *R. P. Agnew* (Ithaca, N.Y.).

Ogieveckiĭ, I. E. Some Tauberian theorems for double series. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 330–333. (Russian)

Several theorems involving Abel and Cesàro transforms of double series employ the definition of restricted evaluability, introduced by C. N. Moore [Math. Ann. **74** (1913) 555–572], and the following definition. A double sequence S_{mn} is said to be restrictedly slowly oscillating if, when $0<\lambda<1$,

$$\lim_{\theta_1,\theta_2} \limsup_{m,n} \max_{j,k} |S_{jk}-S_{mn}|=0$$

where the maximum is taken over $m\leq j\leq(1+\theta_1)m$, $n\leq k\leq(1+\theta_2)^n$; the superior limit is taken as m, $n\to\infty$ subject to the restriction $\lambda n\leq m\leq\lambda^{-1}n$; and the final limit is taken as θ_1, $\theta_2\to 0$ subject to the restriction

$$0<\lambda\theta_1<\theta_2<\lambda^{-1}\theta_1.$$

One of the theorems is the following. Let $\sum u_{mn}$ be a double series which is restrictedly evaluable to S by the Abel method. Let α, $\beta>-1$. Let the Cesàro (C, α, β) transform of $\sum u_{mn}$ be bounded and restrictedly slowly oscillating. Then $\sum u_n$ is restrictedly evaluable (C, α, β) to S. Some of the theorems involve two Cesàro transforms of different orders. *R. P. Agnew* (Ithaca, N.Y.).

Tandori, Károly. Sur les moyennes de Cesàro des séries orthogonales. C. R. Acad. Sci. Paris **244** (1957), 993–995.

Let $\sigma_n^\alpha(x)$, $\bar\sigma_n^\alpha(x)$ denote the Cesàro sums of order $\alpha\geq 0$ of the series $\sum a_n\phi_n(x)$, $\sum \phi_n(x)$, where $\sum a_n^2<+\infty$ and $\phi_n(x)$ is a normed orthogonal system on $[a, b]$. Using theorems of Menchoff and Rademacher the author obtains: (a) $\sigma_n^\alpha(x)=o(\log\log n)$; (b) $\bar\sigma_n^\alpha(x)=\sigma(c_n)$ if $\sum c_n^{-2}<+\infty$ and $c_n/(n\log n)$ increases, and these estimates cannot be improved. Result (a) is stronger than an estimate of Alexits [Ann. Soc. Polon. Math. **25** (1952), 183–187; MR **14**, 1081]; result (b) for $\alpha=1$ was given by Gál [Ann. Inst. Fourier; Grenoble **1** (1949), 53–59, MR **12**, 405]. *G. G. Lorentz* (Ann Arbor, Mich.).

Bosanquet, L. S.; and Chow, H. C. Some remarks on convergence and summability factors. J. London Math. Soc. **32** (1957), 73–82.

If $r>0$, λ_n is nonincreasing and positive, and $\varepsilon_n=O(\lambda_n)$, then $\Delta^r\varepsilon_n=O(n^{-r}\lambda_n)$ if and only if $\Delta^r(n^{-1}\varepsilon_n)=O(n^{-r-1}\lambda_n)$. If $r>-1$, $p\geq 0$, $\sum n^{p-1}|\varepsilon_n|<\infty$, then $\sum n^{p+r}|\Delta^{r+1}\varepsilon_n|<\infty$ if and only if $\sum n^{p+r+1}|\Delta^{r+1}(n^{-1}\varepsilon)|<\infty$. These two theorems are proved and are used to establish equivalence of assertions and theorems about factor sequences for series absolutely evaluable by Cesàro methods. One of the factor sequence theorems characterizes, in terms of real nonnegative orders r and ρ, those sequences ε_n for which $\sum \varepsilon_n a_n$ is evaluable $|C, \rho|$ whenever $\sum a_n$ is evaluable $|C, r|$. *R. P. Agnew* (Ithaca, N.Y.).

Petersen, G. M. Consistent summability methods. J. London Math. Soc. **32** (1957), 62–65.

The author constructs two explicit regular row-finite matrices A and B having real nonnegative elements. The sequence $S^{(1)}=\{1, 0, 1, 0, \cdots\}$ is evaluable A to 1/4 (not 1/2), and the sequences $S^{(2)}=\{1, 0, 0, 0, 1, 0, 0, 0, \cdots\}$ is evaluable B to 1/2 (not 1/4). The two matrices A and B are consistent because they are regular and each sequence (bounded or unbounded) which is both evaluable A and evaluable B must be convergent. Let A^*, B^*, $\cdots$ denote the set of bounded sequences evaluable A, B, $\cdots$. Two facts imply the conclusion that there is no regular matrix C, with real nonnegative elements, for which $C^*\supset A^*$ and $C^*\supset B^*$. In the first place a theorem of Mazur and Orlicz [for references to this theorem (which is sometimes credited to Brudno) see Mazur and Orlicz, Studia Math. **14** (1954), 129–160; MR **16**, 814] implies that if C is regular and $C^*\supset A^*$ and $C^*\supset B^*$, then C must be consistent with A and B. In the second place, if C is regular and C evaluates $S^{(1)}$ to 1/4 and $S^{(2)}$ to 1/2, then the elements of C cannot be real and nonnegative. *R. P. Agnew.*

Kadec, M. I. Unconditionally convergent series in a uniformly convex space. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 185–190. (Russian)

W. Orlicz [Studia Math. **4** (1933), 33–37, 41–47] showed that if $\sum_k x_k$ is an unconditionally convergent series in a space L^p, $p\geq 1$, and if $n=n(p)$, where $n(p)=p$ if $p\geq 2$, $n(p)=2$ if $1\leq p\leq 2$, then $\sum_k \|x_k\|^n$ converges. In this paper it is proved that if B is a uniformly convex space with modulus of convexity $\delta(\varepsilon)$, then $\sum_k \delta(\|x_k\|)$ converges. This gives Orlicz's result when Clarkson's estimate $\delta(\varepsilon)\geq K(p)\varepsilon^{p/(p-1)}$ is improved to $K(p)\varepsilon^2$ in the range $1<p<2$ [Trans. Amer. Math. Soc. **40** (1936), 396–414]. Hanner [Ark. Math. **3** (1956), 239–244; MR **17**, 987] has computed the maximal $\delta(\varepsilon)$ in these spaces; his results also show $\delta(\varepsilon)\geq K(p)\varepsilon^2$ when $1<p<2$. *M. M. Day.*

Davydov, N. A. On the inversion of Abel's theorem. Mat. Sb. N.S. **39(81)** (1956), 401–404. (Russian)

Generalisations of a Tauberian theorem of Evgrafov [Izv. Akad. Nauk SSSR. Ser. Mat. **16** (1952), 521–524; MR **14**, 552] which has as an hypothesis a gap-condition on $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $(|z|<1)$ and the condition (1) $f(z)$ tends to a limit as $z\to 1$ in any manner within the circle $|z-b|<1-b$ $(0<b<1)$. A typical result is as follows. Let $s_n=a_0+a_1\cdots+a_n$. If, for a given sequence $\{n_k\}$ of increasing natural numbers, and for every sequence $\{m_k\}$ of natural numbers such that $m_k>n_k$ and $m_k/n_k\to 1$, we have $s_{m_k}-s_{n_k}\to 0$ as $k\to\infty$, then (1) implies $s_{n_k}\to f(1)$. *W. H. J. Fuchs.*

Meyer-König, W.; und Zeller, K. Lückenumkehrsätze und Lückenperfektheit. Math. Z. **66** (1956), 203–224.

The authors propose a method of reducing the proofs of Tauberian theorems with a gap condition (T): $u_k=0$ for $k\neq k_0, k_1, \cdots$, where $0<k_0<k_1<\cdots$ is a given sequence of integers, to the (much easier) case when the

sequence $s_n=\sum_{k=0}^{n} u_k$ is bounded. If $A=(a_{mk})$ is a regular method of summation, we denote by A^* the method with the coefficients $a^*_{ml}=a_{m k_l A^*}+\cdots+a_{m,k_{l+1}-1}$. A is gap-perfect if A^* for each sequence k_l is perfect, i.e. if finite sequences are dense in the natural topology in the summability field $\mathfrak{A}^*$ of A^*. If A is not gap-perfect, one considers the gap-perfect part $\mathfrak{G}$ of $\mathfrak{A}$ — the set of all sequences x_k in $\mathfrak{A}$ approximable by finite sequences x_k' with the restriction that $x'_{k+1}-x_k'=0$ whenever $x_{k+1}-x_k=0$. The main theorem is that $\mathfrak{G}$ either consists of only convergent sequences, or contains some bounded divergent sequences. This allows the conclusion that if condition (T) is a Tauberian condition for A for bounded sequences, then it is also a Tauberian condition for all sequences in $\mathfrak{G}$. The determination of $\mathfrak{G}$ is thus an essential step in establishing gap Tauberian theorems. The authors show that the methods of Cesàro, C_α, $\alpha\geqq 0$, Abel, and Euler-Knopp E_α, $0\leqq\alpha\leqq 1$ are gap perfect ($\mathfrak{A}=\mathfrak{G}$). For the Borel method B, $\mathfrak{G}$ contains all regularly B-summable sequences s_n (i.e. sequences for which $\sum u_n z^n$ is regular at $z=0$). As an example, the "high indices theorem" for Abel's method $A(\lambda_n)$ is derived from its O-Tauberian theorem. Gap Tauberian theorems of E_α and B are treated in a similar way. *G. G. Lorentz* (Ann Arbor, Mich.).

Gaier, Dieter. Note on some gap theorems. Proc. Amer. Math. Soc. 8 (1957), 24–28.

Let $m_1, m_2, \cdots$ be an increasing sequence of integers and let $\lambda>1$. Let $a_0+a_1+\cdots$ be a series satisfying the Tauberian gap condition $a_m=0$ when $m_k<m\leqq\lambda m_k$. Let the series in $f(z)=\sum a_m z^m$ converge when $|z|<1$ and define a function $f(z)$ which is bounded over some circle $|z-\alpha|<1-\alpha$ for which $0<\alpha<1$. Let $\lim_{x\to 1-} f(x)=s$. Then $\sum_{m=0}^{m_k} a_m\to s$ as $k\to\infty$. Related theorems and implications, including known theorems with stronger hypotheses, are given. *R. P. Agnew* (Ithaca, N.Y.).

See also: Stöhr, p. 712; Carlitz, p. 717; San Juan, p. 724; Karadžić, p. 725; Ghizzetti, p. 735; Sharma and Srivestava, p. 736; Žautykov, p. 740; Marhasev, p. 743; Jerison, p. 747; Wigner, p. 771.

Approximations, Orthogonal Functions

Vertgeim, B. A. On conditions of applicability of Newton's method. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 719–722. (Russian)

This paper extends some results of L. V. Kantorovič [Uspehi Mat. Nauk (N.S.) 3 (1948), no. 6(28), 89–185; MR 10, 380] on the application of Newton's method for the approximate solutions of functional equations of the form $P(x)=0$, where $P(x)$ is a nonlinear twice differentiable (in the sense of Fréchet) operator between Banach spaces. Here the assumption of twice differentiability of $P(x)$ is replaced by the hypothesis that $P'(x)$ satisfies the Hölder condition

$$\|P'(x_1)-P'(x_2)\|\leqq K\|x_1-x_2\|^\alpha,$$

where K and α are constants and $0<\alpha\leqq 1$.
H. P. Thielman (Ames, Iowa).

Mergelyan, S. N. Weighted approximations by polynomials. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 5(71), 107–152. (Russian)

A very clear account of weighted approximation by polynomials on the whole real axis and on unbounded, nowhere dense, closed sets in the z-plane. [Mergelyan, Dokl. Akad. Nauk SSR (N.S.) **97** (1954), 597–600; Akad. Nauk Armyan. SSSR Dokl. **20** (1955), 113–119; MR **16**, 1104; **17**, 148; Děrbašyan, Mat. Sb. N.S. **36(78)** (1955), 353–440; MR **17**, 31; Šaginyan, Akad. Nauk. Armyan. SSR. Dokl. 3 (1945), 33–38]. Although most of the results were published before in the papers just cited and the literature quoted in them, this exposition contains many simplifications and additions. As an example I quote: If $0\leqq h(x)\leqq 1$ $(-\infty<x<\infty)$, define $M(h(x))$ as the class of polynomials $P(x)$ such that $h(x)|P(x)|\leqq 1$ $(-\infty<x<\infty)$. Let $C(h)$ be the space of continuous functions $f(x)$ such that $h(x)f(x)\to 0$, as $|x|\to\infty$, with the norm

$$\|f\|=\sup h(x)|f(x)|\ (-\infty<x<\infty).$$

The polynomials are fundamental (= closed) in $C(h)$, if and only if $\int_{-\infty}^{\infty}(\sup\log|P(x)|)dx/(1+x^2)=\infty$, where the sup is taken over all $P\in M(h(x)/(1+|x|))$.
W. H. J. Fuchs (Ithaca, N.Y.).

Ghermănescu, M. Suites orthogonales, invariantes par dérivation. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 8 (1956), 537–547. (Romanian. Russian and French summaries)

If the set of functions $\{P_n(x)\}$ is orthonormal over $[a, b]$ with respect to some weight function, then, in general, the set of derivatives is not orthogonal. In the present paper conditions are investigated, that are either necessary, or sufficient for the orthogonality of the set $\{P_n'(x)\}$, or, more generally, of $\{P_n^{(k)}(x)\}$, with respect to some weight function. Let $P_n(x)$ be square integrable, of class C^2 and orthonormal with respect to $p(x)$; then $P_n(x)$ satisfies a differential equation

$$(*)\qquad A(x)P_n''(x)+B(x)P_n'(x)-\lambda_n P_n(x)=0,$$

with given constants λ_n, $A(x)>0$ in (a, b) and

$$A(x)p(x)=\exp\left(\int_a^x\frac{B(x)}{A(x)}dx\right).$$

Set $q(x)=A(x)p(x)$; then the derivatives $P_n'(x)$ are orthogonal with respect to $q(x)$. In order to maintain the orthogonality indefinitely under successive differentiation, a necessary condition is that $A(x)$ should be a quadratic and $B(x)$ a linear polynomial. It is shown that most classical orthogonal polynomials and their associated functions maintain their orthogonal character under repeated differentiation. Also a more general theorem is given, that takes into account also the case of incomplete orthonormal sets $\{P_n(x)\}$. *E. Grosswald.*

Tandori, Károly. Quelques évaluations sur les fonctions orthogonales. C. R. Acad. Sci. Paris **244** (1957), 836–838.

Let $\{\phi_n(x)\}$ be an orthonormal set of functions on $[a, b]$. If $\sum a_n^2<\infty$, then

$$(1)\qquad \sum_0^N a_n\phi_n(x)=o(\log N)$$

almost everywhere on $[a, b]$. If $\nu(n)$ is a positive increasing sequence such that

$$\sum_2^\infty (n\log n)^{-1}\nu^{-2}(n)<\infty,$$

then

$$(2)\qquad \sum_0^N \phi_n(x)=o(N^{1/2}(\log N)^{3/2}\nu(n)),$$

$$(3)\qquad \sum_0^N \phi_n^2(x)=o(N\log N\,\nu^2(N))$$

for almost all x on $[a, b]$. The author announces that the estimates (1), (2), (3), which are due to Rademacher [Math. Ann. 87 (1922), 112–138] and Kaczmarz [Studia Math. 1 (1929), 87–121], cannot be improved. The estimates (1) and (2) are best possible even for uniformly bounded sets $\{\phi_n(x)\}$. *W. Rudin* (Rochester, N.Y.).

Geronimus, Ya. L. On certain sufficient conditions for convergence of the Fourier-Čebyšev process. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 907–909. (Russian)

Let $f(x)$ be defined on $[-1, 1]$; let $p_n(x)$ be orthogonal polynomials with weight $d\psi(x)$; the "Fourier-Čebyšev process" is the expansion of f in its Fourier series with respect to the p_n. The author tabulates 9 sets of conditions bearing jointly on f and ψ which suffice for uniform convergence either on the whole interval or in subintervals. Let $t(x)=\psi'(x)(1-x^2)^{\frac{1}{2}}$, let $\omega(\delta; f)$ denote the modulus of continuity of f on $[a, b]\subset[-1, 1]$, and let $\omega_2(\delta; f)$ denote the L^2 modulus of continuity on $[-1, 1]$. Some sample sets of conditions are (for uniform convergence on $[a, b]$) as follows: (II) ψ is absolutely continuous no $[-1, 1]$, $t(x)\leq M$, $\log t(\cos\phi)\in L[-1, 1]$, and $f(\cos\phi)\in L^2(-1, 1)$; and in $[a, b]$, t is continuous, $\omega(\delta; t)=O(\delta^\alpha)$, $\alpha>\frac{1}{2}$; $0<m\leq t(x)$; and $\omega(\delta; f)=O(-1/\log\delta)$; (VI) $\psi(x)$ is absolutely continuous on $[-1, 1]$, $0<m\leq t(x)\leq M$, $f(\cos\phi)\in L^2(-1, 1)$, $\omega_2(\delta; f)\omega_2(\delta; t)=o(\delta^{\frac{1}{2}})$ on $(-1, 1)$; and on $[a, b]$, f is continuous and $\omega(\delta; f)\omega_2^2(\delta; t)=o(-\delta/\log\delta)$. *R. P. Boas, Jr.* (Evanston, Ill.).

See also: San Juan, p. 724; R.-Salinas, p. 724; Trošin, p. 727; Leont'ev, p. 728; Al'per, p. 728; Ibragimov, p. 729; Voronovskaya, p. 730; Tandori, p. 733; Kantorovič, p. 747.

Trigonometric Series and Integrals

Ghizzetti, Aldo. Sui coefficienti di Fourier-Stieltjes di una funzione non decrescente. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 580–583.

The author gives a simple proof of the well-known necessary and sufficient conditions for a set of numbers $\{c_k\}$ to be the trigonometric moments of a nondecreasing function, the novelty being in his proof of the sufficiency. The condition is that the forms $H_n=\sum\sum_0^n c_{k-h}x_k\bar{x}_h$ are non-negative. Put $\lambda_n=(n+1)^{-\frac{1}{2}}$; the author takes

$$\sigma_n(t)=H_n(\lambda_n, \lambda_n e^{it}, \cdots, \lambda_n e^{int})\geq 0,\ \alpha_n(t)=\int_0^t \sigma_n(u)du,$$

and shows by using Helly's theorem that $\{\alpha_n\}$ contains a subsequence converging to a solution of the moment problem. *R. P. Boas, Jr.* (Evanston, Ill.).

Satô, Masako. Fourier series. II. Order of partial sums. Proc. Japan Acad. 32 (1956), 529–534.

$s_n(x)$ denotes the nth partial sum of the Fourier series of an L-integrable $f(x)$, and $\varphi_x(u)=f(x+u)+f(x-u)$. Let

$$I_1(t)=\int_0^t \varphi_x(u)du,\ I_2(t)=\int_0^t |\varphi_x(u)|du,$$

$$I_3(t,\xi)=\int_0^t \{f(\xi+u)-f(\xi-u)\}du.$$

It is a classical result of Lebesgue that $I_2(t)=o(t)$ implies $s_n(x)=o(\log n)$. $I_1(t)=o(t)$ and $I_2(t)=O(t)$ do not imply this [S. Izumi, Tôhoku Math. J. (2) 1 (1950), 144–166; MR 11, 656]. It is now proved that $I_1(t)=o(t)$ and $I_3(t, \xi)=o(t)$, uniformly in ξ near x, imply $s_n(x)=o(\log n)$. Similar results hold with respect to $s_n(x)=o(\log\log n)$; $o(t)$ is replaced by $o(t/\log 1/t)$. [Cf. O. Szász, Bull. Amer. Math. Soc. 48 (1942), 705–711; MR 4, 37.]
W. W. Rogosinski (Newcastle-on-Tyne).

Lorch, Lee. The Gibbs phenomenon for Borel means. Proc. Amer. Math. Soc. 8 (1957), 81–84.

The Borel exponential and integral means of the Fourier series $\sum n^{-1}\sin nx$ are shown to display the same Gibbs phenomenon and to possess the same Gibbs ratio as that exhibited by the partial sums of the series.
P. Civin (Eugene, Oreg.).

Koizumi, Sumiyuki. Correction and remark on the paper: On integral inequalities and certain of its applications to Fourier series. This journal Vol. 7 (1955), No. 1–2, pp. 119–127. Tôhoku Math. J. (2) 8 (1956), 235–243.

Let $Q(z)$ be regular in the unit circle, of class H^r $(r>1)$ and with boundary function $Q(e^{i\theta})$. Let

$$Q_s^*(\theta)=\sup_{0<|h|<\pi}\left(\frac{1}{h}\int_0^h |Q(e^{i\theta+it})|^s dt\right)^{1/s}.$$

The author amends Theorem D of his earlier work [same J. (2) 7 (1955), 119–127; MR 17, 361] to read for $r>1$ and $2\leq q<2r$,

$$\int_{-\pi}^{\pi}(g^*_q(\theta))^r d\theta\leq A_{r,q}\int_{-\pi}^{\pi}|Q(e^{i\theta})|^r d\theta.$$

Further extension is made of the above result and application is given to the strong summability of Fourier series. *P. Civin* (Eugene, Ore.).

See also: Ul'yanov, p. 726; Geronimus, p. 735; Ehrenpreis, p. 746.

Integral Transforms

Doetsch, Gustav. Stabilitätsuntersuchung von Regelungsvorgängen vermittels Laplace-Transformation. Österreich. Ing.-Arch. 10 (1956), 140–148.

Expository paper. *G. E. H. Reuter* (Manchester).

Toll, John S. Causality and the dispersion relation: logical foundations. Phys. Rev. (2) 104 (1956), 1760–1770.

Let $F(t)$ and $G(t)$ denote input and output for a system subject to a superposition principle such that

$$G(t)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}T(t-t')F(t')dt',$$

or in terms of Fourier transforms,

$$g(\omega_r)=A(\omega_r)f(\omega_r).$$

The author is concerned with the purely mathematical problem of characterizing A so that G satisfies the condition of strict causality: (i) $G(t)=0$ for $t<0$ when $F(t)=0$ for $t<0$ (no output before the input). Under the very general conditions that F is square-integrable and $|A(\omega_r)|$ is bounded, it is shown that (i) is equivalent to each of six conditions on A which may be briefly indicated as follows: (ii) $A(\omega_r)$ is the boundary value function for a most all real ω_r of a function which is analytic and of

absolute value less than unity throughout the upper half of the complex ω plane. (This "regularity" condition should be compared with the familar definition of a "causal transform".) (iii) [iv] $A(\omega_r)$ is expressible in terms of its real [imaginary] part. The author points out various physical interpretations of the integrals in these "dispersion" relations.) (v), (vi) Certain Cauchy-type integrals of A vanish for points in the lower half-plane. (vii) For some $\bar{\omega}$ with $\bar{\omega}_i<0$, $A(\nu)/(\nu-\bar{\omega})$ is a causal transform. A bounded measurable function $A(\omega_r)$ satisfying these conditions is called a "causal factor", for its product with any causal transform is another causal transform.

The author puts the dispersion relation into phase-shift form, and considers the effect of relaxing the condition of boundedness on A for the cases where $A(\omega_r)$ diverges at ∞ or at a finite point. *C. C. Torrance.*

Gel'fand, I. M.; and Šilov, G. E. Fourier transforms of rapidly increasing functions and questions of the uniqueness of the solution of Cauchy's problem. Amer. Math. Soc. Transl. (2) **5** (1957), 221–274.

A translation of the authors' 1953 Russian paper reviewed in MR **15**, 867.

Sharma, A.; and Srivastava, H. M. On certain functional relations and a generalization of the $M_{k,m}$ function. Ann. Polon. Math. 3 (1956), 76–86.

R. P. Agarwal [Ann. Soc. Sci. Bruxelles. Sér. I. **64** (1950), 164–168; Bull. Calcutta Math. Soc. **43** (1951), 153–167; **45** (1953), 69–73; MR **12**, 605; **14**, 639; **15**, 524] generalized the Hankel transformation, replacing the Bessel function by a function derived from the series

$$\sum_{r=0}^{\infty} \frac{(-x)^r}{r!\Gamma(\nu+\lambda r+1)}$$

where $\lambda>0$ ($\lambda=1$ for Bessel functions). For the sogeneralized Hankel transform of a fixed function, the authors derive recurrence and other relations. In this connection they define, and obtain formulas for, a generalization of the confluent hypergeometric function and generalized Laguerre polynomials. The latter are generated when

$$(1-u)^{-\nu-1} \exp\{x-x(1-u)^{-\lambda}\}$$

is expanded in powers of u. *A. Erdélyi.*

See also: Bergman, p. 741; Miyadera, p. 748; Lukacs, p. 769; Ritchie and Sakakura, p. 780.

Ordinary Differential Equations

Slugin, S. N. Application of a Čaplygin type method of approximate solution of operator equations. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 739–741. (Russian)

The note illustrates the application of a general theorem [same Dokl. (N.S.) **103** (1955), 565–568; MR **17**, 387]. Four examples are given.

Example 1. Consider the differential equation

$$(1) \qquad y'-f(t,y)=0, \quad y(t_0)=\alpha.$$

If $\underset{\sim}{y}_0$ and $\bar{y}_0$ are functions satisfying, for $t\geqq t_0$, $\underset{\sim}{y}_0(t)\leqq\bar{y}_0(t)$, $\underset{\sim}{y}_0'-f(t,\underset{\sim}{y}_0)\leqq 0\leqq \bar{y}_0'-f(t,\bar{y}_0)$, and $\underset{\sim}{y}_0(t_0)=\bar{y}_0(t_0)=\alpha$; if $f(t,y)$ is continuous in t and if $C>\partial f/\partial y\geqq -A$ $(C>0)$ for all functions $y(t)$ $(\underset{\sim}{y}_0\leqq y\leqq\bar{y}_0)$; then the iterative procedure

$$y_{n+1}=\alpha e^{A(t_0-t)}+\int_{t_0}^{t}[Ay_n(\tau)+f(\tau,y_n(\tau))]e^{A(\tau-t)}d\tau$$

yields upper and lower approximations that converge to the unique solution of (1). The rate of convergence is at least that of an exponential series.

Example 2. The above process is generalized for an n-dimensional system.

Example 3. Consider the integral equation

$$(2) \qquad y(t)-\int_{t_0}^{t}K(t,\tau,y^{(\tau)})d\tau=0.$$

K is assumed to be continuous in t, and $0\leqq\partial K/\partial y\leqq B$. Again, starting with upper and lower zero approximations, the iterative procedure

$$y_{n+1}=\int_{t_0}^{t}K(t,\tau,y_n(\tau))d\tau$$

yields upper and lower approximations that converge to the unique solution of (2) at least as rapidly as an exponential series.

Example 4. Consider the system of equations

$$(3) \qquad f_k(t_1,\cdots,t_m)=0 \quad (k=1,\cdots,m)$$

where the f_k are real-valued functions of a real variable. Assume

$$f_k(\underset{\sim}{t}_{10},\cdots,\underset{\sim}{t}_{m0})\leqq 0\leqq f(\bar{t}_{10},\cdots,\bar{t}_{m0}) \quad (k=1,\cdots,m);$$

for all $t_k(\underset{\sim}{t}_{k0}\leqq t_k\leqq\bar{t}_{k0})$ $\partial f_i/\partial t_k\leqq a_{ik}$; $\det\|a_{ik}\|>0$; and $A_{ik}\geqq 0$ where A_{ik} is the cofactor of a_{ik}. Here the iterative procedure that yields upper and lower approximations converging to a solution of (3) is

$$t_{k(n+1)}=t_{kn}-\sum_{i=1}^{m}\frac{A_{ik}f_k(t_{1n},\cdots,t_{mn})}{\det\|a_{ik}\|}.$$

J. P. LaSalle (Notre Dame, Ind.).

Burton, L. P. Conditions which preclude the existence of critical solutions of an ordinary differential system. Proc. Amer. Math. Soc. 7 (1956), 791–795.

L'A. considera il sistema di equazioni

$$y_{i+1}'=f_{i+1}(x,y_i) \quad (i=1,2,\cdots,n;\ y_{n+1}=y_1)$$

nell'ipotesi che un numero dispari delle f_{i+1} siano funzioni descrescenti di y_i e le rimanenti crescenti. Se $\{y_i(x)\}$ e $\{z_i(x)\}$ sono due soluzioni del sistema definite nell'intervallo $x_0<x<x_0+a$, la prima verificante le condizioni iniziali: $y_i(x_0)=y_{i0}$, la seconda tale che il punto $(x_0, y_{10},\cdots,y_{n0})$ sia di accumulazione per il suo diagramma, è impossibile che in ogni intervallo $x_0<x<x_0+b$ con $b<a$ sia per ogni k: $z_k<y_k$. *C. Miranda* (Napoli).

Conti, Roberto. Sulla prolungabilità delle soluzioni di un sistema di equazioni differenziali ordinarie. Boll. Un. Mat. Ital. (3) **11** (1956), 510–514.

Consider the system of differential equations

$$(1) \qquad \dot{x}=f(t,x) \quad (a<t<b),$$

where $f(t,x)$ is a real continuous function defined in S: $a<t<b$, $0\leqq\sum x_i^2<+\infty$. Theorem: Let $\omega(t,u)$ be a real continuous function in S_1: $a<t<b$, $0<u<+\infty$; let $u_0(t)$ be the maximum solution of the equation (2) $\dot{u}=\omega(t,u)$, $u(t_0)=u_0$ and let T^+ be the upper limit of the values of t for which $u_0(t)$ is defined. Let $V(t,x)$ be a real, continuous, non-negative function defined in S and having a continuous first derivative in S. Suppose $V_x(t,x)f(t,x)+V_t(t,x)\leqq\omega(t,V(t,x))$ at all points $(t,x)\in S$, $V(t_0,x^0)=u_0$, and $\lim_{\|x\|\to+\infty}V(t,x)=+\infty$, $\|x\|=(\sum x_i^2)^{\frac{1}{2}}$. If $T^+=b$ and the above hypotheses are satisfied, then the solution of system (1) with initial conditions $x(t_0)=x_0$ can

be extended in t. This theorem generalizes a previous result of the author [same Boll. (3) **11** (1956), 344–349; MR **18**, 309]. *J. K. Hale* (Minneapolis, Minn.).

Urabe, Minoru. Application of majorized group of transformations to ordinary differential equations with periodic coefficients. J. Sci. Hiroshima Univ. Ser. A. **19** (1956), 469–478.

A new proof of a theorem of Sibuya [J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1954), 19–32; MR **15**, 872] based on previous results of the author [J. Sci. Hiroshima Univ. Ser. A. **16** (1952), 267–283; MR **16**, 142]. *J. L. Massera.*

Mlak, W. On the epidermic effect for ordinary differential inequalities of the first order. Ann. Polon. Math. **3** (1956), 37–40.

L'A. amplia leggermente la portata di una nota condizione, sufficiente affinché per le funzioni $\varphi_1(t), \cdots, \varphi_n(t)$, continue nell'intervallo $t_0 \leqq t \leqq t_0+a$, si abbia ivi $\varphi_1(t) \leqq \tau_1(t), \cdots, \varphi_n(t) \leqq \tau_n(t)$, qualora sia $\varphi_1(t_0) \leqq \tau_1(t_0), \cdots, \varphi_n(t_0) \leqq \tau_n(t_0)$ e le funzioni $\tau_1(t), \cdots, \tau_n(t)$ forniscano, nell'intervallo $t_0 \leqq t \leqq t_0+a$, un integrale massimo destro per un sistema differenziale del tipo $y_1'=f_1(t, y_1, \cdots, y_n), \cdots, y_n'=f_n(t, y_1, \cdots, y_n)$. Come è noto, e come l'A. ricorda, se $n>1$ questo integrale massimo si definisce soltanto sotto opportune condizioni per le funzioni f.

G. Scorza-Dragoni (Padova).

Petropavlovskaya, R. V. On the oscillatory aspect of solutions of the differential equation $u''=f(u, u', t)$. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 389–391. (Russian)

In this paper all variables are real, and all functions are real and continuous. A solution $u(t)$ of (1) $u''=f(u, u', t)$ with $f \in C$ for all u and u' and $t>a$, is called regular if it can be extended to $t=+\infty$. A regular solution is called oscillatory if it has a sequence $\{t_n\}$ of zeros with $t_n \to +\infty$ as $n \to \infty$. A solution of (1) which can be extended only to $T<\infty$ is called irregular. An irregular solution is called oscillatory if it has a sequence $\{t_n\}$ of zeros with $t_n \to T-0$ as $n \to \infty$.

The author states without proofs three theorems and nine corollaries. Theorem 1: Consider equation (2) $v''=g(v, v', t)$ with $u^{-1}f<v^{-1}g$ for all $t>a$, $0<|u| \leqq |v|$ when $uv>0$, and $|u'| \leqq |v'|$. If $u(t)$ and $v(t)$ are solutions of (1) and (2) respectively on some open interval including t_0, $u \neq 0$ on this interval, and $u=v$, $u'=v'$ at $t=t_0$, then $|u|<|v|$ on the interval except at t_0. Among the corollaries to the theorem are that if every solution of (2) is oscillatory then every regular solution of (1) is oscillatory, and that if $p(t)$ and $q(t)$ are such that $p<u^{-1}f<q$ for $t>a$, and if all solutions of $u''=qu$ are oscillatory, then all solutions of (1) are regular and oscillatory. Theorem 2: In (3) $u''+u\varphi(u, t)=0$ $(t>a)$, let 1) $\varphi>0$ for $u \neq 0$, 2) φ increase with $|u|$, and either 3) every solution of $u''+u\varphi(u_0, t)=0$ is oscillatory for every $u_0 \neq 0$ or 3') every solution of $u''+u\varphi(c, t)=0$ is oscillatory for some $c \neq 0$ and for every $u_0 \neq 0$ $\int_{a+1}^{\infty} dz \int_z^{\infty} \varphi(u_0, t)dt=\infty$. Then every solution of (3) is oscillatory. A result of Wintner [Quart. Appl. Math. **7** (1949), 115–117; MR **10**, 456] is combined with this theorem to give that all solutions of $u''+L(u)p(t)=0$ are oscillatory if $u^{-1}L(u)>0$ for $u \neq 0$ and increases with $|u|$, and $p(t)>0$ for $t>a$ with $\int_{a+1}^{\infty} p(t)dt=\infty$. Theorems 1 and 2 are combined to give Theorem 3, which is a condition for oscillation of solutions of $u''+u[\varphi(u, t)+\psi(u, u', t)]=0$.

W. S. Loud (Minneapolis, Minn.).

Corduneanu, C. Sur un problème aux limites concernant les équations différentielles non-linéaires du second ordre. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 11–16. (Romanian. Russian and French summaries)

Let $f(x, u, v)$ be continuous and possess continuous partial derivatives f_u, f_v in the slab $a \leqq x \leqq b$, $-\infty<u$, $v<\infty$; let $f_v \leqq 0$, $|f_u|<2\{(b-a)^2+2|h|(b-a)\}^{-1}$ in the slab, and let f_v be bounded below in $a \leqq x \leqq b$, $-M \leqq u \leqq M$, $-\infty<v<\infty$ for each M. The author proves that the nonlinear differential equation $y''=f(x, y, y')$ possesses a unique solution in $a \leqq x \leqq b$ satisfying the boundary conditions $y'(a)=A$, $y(b)+hy'(b)=B$, and shows the construction of this solution by successive approximations. {It seems to the reviewer that $h \geqq 0$ must be assumed for the general truth of the statement.}

A. Erdélyi (Jerusalem).

Wintner, Aurel. A criterion for homogeneous linear differential equations with damped solutions. J. Math. Mech. **6** (1957), 109–117.

A real- or complex-valued function, defined for large positive t, is called exponentially small if it is $O(e^{-\gamma t})$ as $t \to \infty$, where γ is some positive constant. The author proves that the following four statements are equivalent: (I) If $\omega=\omega(r)$, $0<r \leqq r_0$, is any continuous function for which $r\omega(r)$ stays between two positive bounds as $r \to 0$, then there exists a constant $\alpha=\alpha(\omega)$ such that (1) $x(r)=O(r^\alpha)$ as $r \to 0$, holds for every solution $x(r)$ of (2) $d^2x/dr^2+\omega^2x=0$. (II) If $\varphi(r)$, $\psi(r)$, $0<r \leqq r_0$, are real-valued, continuous functions which are bounded as $r \to 0$ and if $\varphi(r)$ stays above a positive bound and $\psi(r)$ stays below a bound which is less than 1, then there exists a constant $\alpha=\alpha(\psi, \varphi)>0$ such that (1) holds for every solution $x(r)$ of (3) $r^2(d^2x/dr^2)+\psi(r)r(dx/dr)+\varphi(r)x=0$. (III) If $g=g(t)$, $f=f(t)$, $t_0 \leqq t<\infty$, are continuous functions which stay between positive bounds at $t \to \infty$, then all solution $x(t)$ of (4) $d^2x/dt^2+gdx/dt+fx=0$ are exponentially small. (IV) If $g=1$ and $f(t)$ stays between positive bounds as $t \to \infty$, then all solutions of (4) are exponentially small. *J. K. Hale* (Minneapolis, Minn.).

Prachar, K.; und Schmetterer, L. Über eine spezielle nichtlineare Differentialgleichung. Österreich. Ing.-Arch. **10** (1956), 247–252.

Let f be continuous, odd and increasing on $(-\infty, \infty)$, with $f(+\infty)=1$, $0<f(v)<v$ for $v>0$, $f(v)=v+O(v^\alpha)$ as $v \to +0$ for some $\alpha>1$. Then the system $v'+2x^{-1}v=y$, $y'=f(v)$ has a solution, defined for $x>0$, such that $y(x) \sim x^{-1}e^{-x}$ and $v(x)+x^{-1}e^{-x} \sim -x^{-2}e^{-x}$ as $x \to +\infty$. This is proved by using an appropriate iterative method. The special choice $f(v)=v(1+v^2)^{-\frac{1}{2}}$ leads to the equation $y''+2x^{-1}(1-y'^2)y'=y(1-y'^2)^{3/2}$. *G. E. H. Reuter.*

★ **Swanson, C. A. Differential equations with singular points.** Tech. Rep. 16. Department of Mathematics, California Institute of Technology, Pasadena, 1956. 24 pp.

The author presents a refinement of results of R. E. Langer [Trans. Amer. Math. **37** (1935), 397–416] which pertain to the differential equation

$$d^2u/dx^2+[\lambda^2q(x)+r(x, \lambda)]u=0.$$

The author's results are obtained under the principal assumptions that x is a real variable and that

$$q(x)=(x-c)^{-1}Q(x),$$

where $Q(x) \in C^2$ and is nonvanishing.

N. D. Kazarinoff (Ann Arbor, Mich.).

Kimura, Tosihusa. Sur les points singuliers essentiels mobiles des équations différentielles du second ordre. Comment. Math. Univ. St. Paul. 5 (1956), 81–94.

If $R(z, w)$ is a rational function of w and an analytic function of the complex variable z, the mobile singularities of the differential equation $dw/dz=R(z, w)$ are known to be algebraic. In the case of a second-order equation $w''=R_1(z, w, w')$ this is in general not true, and the mobile singularities may be of a transcendental character. The author studies the properties of singularities of the latter type if the equation is of the particular form $w''=P(z, w, w')\,[\theta(z, w, w')]^{-1}$, where both P and Q are polynomials in w and w', and he obtains a number of results concerning the behavior of solutions at such points.

Z. Nehari (Pittsburgh, Pa.).

Langer, Rudolph E. On the asymptotic solutions of a class of ordinary differential equations of the fourth order, with special reference to an equation of hydrodynamics. Trans. Amer. Math. Soc. 84 (1957), 144–191.

In the differential equation

$$(1)\quad \mathscr{L}(w)\equiv w^{(4)}+\lambda^2\{P(z, \lambda)w''+Q(z, \lambda)w'+R(z, \lambda)w\}=0$$

let P, Q and R be regular analytic functions of z and λ for $|z|\leqq z_0$, $|\lambda|\geqq\lambda_0$. If $P(0, \infty)=0$, while $[\partial P(z, \infty)/\partial z]_{z=z_0}\neq 0$, the differential equation is said to have a turning point of first order at $z=0$. Under these assumptions the author derives asymptotic expressions, as $\lambda\to\infty$, for fundamental systems of (1). The approximations are uniformly valid in certain sectors of the complex z and λ planes. Every pair of values of $\arg z$ and $\arg\lambda$ lies in some pair of "associated regions" for which such a fundamental system can be found, and every z-region contains the point $z=0$. The relative error of these approximations is $O(\lambda^{-m_1})$, where m_1 can, in most cases, be chosen arbitrarily.

The construction of the asymptotic solutions is rather complicated and can only be sketched briefly here. The starting point is the result of a previous paper by the author [Duke Math. J. 23 (1956), 93–110; MR 18, 127] in which a similar asymptotic theory for differential equation of the form

$$(2)\quad L(u)\equiv u'''+\lambda^2p(z, \lambda)u'+\lambda^2q(z, \lambda)u=0$$

is developed. Every solution of (2) satisfies also the fourth order differential equation

$$\frac{d}{dx}L(u)+\lambda^{-m-1}r(z, \lambda)L(u)=0.$$

By means of a suitable sequence of algorithms the functions $p(z, \lambda)$, $q(z, \lambda)$, $r(z, \lambda)$ and two more functions $\mathscr{A}(z, \lambda)$ and $\mathscr{C}(z, \lambda)$ are determined in such a way that the functions $\eta=\mathscr{A}(z, \lambda)u+\lambda^{-2}\mathscr{C}(z, \lambda)u''$ satisfy a "related" differential equation

$$(3)\quad \mathscr{L}(\eta)+\lambda^{-m-1}\mathscr{M}(\eta)=0,$$

where $\mathscr{M}(\eta)$ is a linear differential expression of third order. By means of a classical comparison technique it is then shown that (3) possesses solutions that differ by terms of order $O(\lambda^{-m_1})$ from solutions of the given equation (1).

There is a certain special category of differential equations of type (1) to which the author's arguments do not apply. This exceptional category is a subclass of the differential equations for which the "reduced equation"

$$P(z, \infty)w''+Q(z, \infty)w'+R(z, \infty)w=0$$

has at $z=0$ characteristic exponents that differ by an integer. It is shown that the Sommerfeld-Orr equation for small disturbances of a laminar flow can in general — but not always — be asymptotically solved by the method of this paper.

W. Wasow (Madison, Wis.).

Pliś, A. Remarque sur le système dynamique dans le domaine doublement connexe. Ann. Polon. Math. 3 (1956), 169–171.

This paper is in the line of the Ważewski's topological theory [see e.g. T. Ważewski, same Ann. 1 (1955), 338–345; MR 17, 611; Z. Mikołajska, ibid. 1 (1955), 277–305; MR 17, 615; A. Plis, Bull. Acad. Polon. Sci. Cl. III. 2 (1954), 415–418; MR 16, 700; F. Albrecht, ibid. 2 (1954), 315–318; MR 16, 248] and concerns the differential system (*) $dx/dt=X(x, y)$, $dy/dt=Y(x, y)$, where X, Y are continuous functions in the whole xy-plane and a uniqueness theorem holds. An annular region G is considered lying between two simple closed curves C_1, C_2, and the following hypotheses are made concerning the set S of the points of egress from G: all points of S are points of strict egress; $S\neq C_1$, $S\neq C_2$, $S\neq 0$, $S\neq C_1+C_2$. Under these hypotheses there exists in G at least one singular point for (*); i.e., a point where $X=Y=0$.

L. Cesari.

Demidovič, B. P. On bounded solutions of a certain nonlinear system of ordinary differential equations. Mat. Sb. N.S. 40(82) (1956), 73–94. (Russian)

The author presents a number of results concerning the existence, uniqueness and stability of solutions of equations having the general form $dx/dt=Ax+g(x, t, \mu)$, where A is a constant matrix and μ is a small parameter, under various assumptions concerning the function g and the matrix A. A number of results pertain to the existence of almost-periodic solutions, related to the earlier work of Bohl [Bull. Soc. Math. France 38 (1910), 5–138]. The techniques used are partly analytic and partly dependent upon a topological principle of Ważewski [Ann. Soc. Polon. Math. 20 (1947), 279–313; MR 10, 122].

R. Bellman (Santa Monica, Calif.).

Mikołajska, Z. Sur les transformations des systèmes d'équations différentielles linéaires aux coefficients variables. Ann. Polon. Math. 3 (1956), 142–146.

Se la trasformazione $x_i=F^i(\tau, \xi_1, \cdots, \xi_n)$, $t=\tau$ $(i=1, \cdots, n)$ è ovunque invertibile nello spazio $(\tau, \xi_1, \cdots, \xi_n)$, le funzioni F^i riuscendovi continue insieme colle loro derivate parziali prime (di guisa che il loro determinante funzionale rispetto alle ξ è ovunque diverso da zero), e se la trasformazione inversa, applicata ad un sistema qualunque di equazioni differenziali ordinarie lineari tutte del primo ordine, porta ad un sistema dello stesso tipo, la trasformazione data è essa stessa lineare.

G. Scorza-Dragoni (Padova).

★ **Amerio, Luigi. Bounded or almost-periodic solutions of non-linear differential systems.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 179–182. University of Maryland Book Store, College Park, Md., 1956.

Summary of recent results of the author [Ann. Mat. Pura Appl. (4) 39 (1955), 97–119; MR 18, 128].

G. E. H. Reuter (Manchester).

Matthies, Karl. Bedingungen für gleichmässige Stetigkeit bzw. Stabilität der Lösungen gewisser Differentialgleichungssysteme. Arch. Math. **7** (1956), 349–353.

Consider the equation

$$dx/dt=f(t, x), \tag{1}$$

where $x=(x_1, \cdots, x_n)$, $f(t, 0)=0$ and $f(t, x)$ is analytic in x and continuous in t in a region R of t, x space. Using the theory of complex variables, the author discusses the region of convergence of the Liapounoff series [Problème général de la stabilité du mouvement, Princeton, 1947, Ch. I, § 2; MR **9**, 34] for the solutions of (1). He also discusses the uniform continuity and the stability of the solutions. No quantitative results are given concerning the size of the region of convergence.

J. K. Hale (Minneapolis, Minn.).

Garcia, Godofredo. On non-linear differential equations. Actas Acad. Ci. Lima **19** (1956), 27–50. (Spanish)

Reproduces some of the calculations of Krylov and Bogoliubov [Introduction to non-linear mechanics, Princeton, 1943; MR **4**, 142], mostly about van der Pol's equation, and with all explanations omitted.

S. Lefschetz (Mexico, D.F.).

Benz, G. Die mechanische Bedeutung des instabilen Zweiges der Frequenz-Amplituden-Kurve bei parametererregten Schwingungen. Z. Angew. Math. Mech. **36** (1956), 273–274.

Haimovici, Adolf. Sur quelques problèmes aux limites non linéaires et sur une représentation des fonctions. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 1–10. (Romanian. Russian and French summaries)

The author discusses the non-linear boundary value problem $y''=f(x, y, y', k)$, $y'(a)=V_1(y(a))$, $y'(b)=V_2(y(b))$, where it is assumed that for each value of the parameter k the differential equation possesses a "general solution" which is continuous and bounded on the interval $a\leqq x\leqq b$ and is differentiable with respect to the constants of integration. $V_1(z)$ and $V_2(z)$ are functions whose reciprocals are integrable over the interval determined by the bounds of the solution of the differential equation.

A. Erdélyi (Jerusalem).

Cimino, Massimo. Una condizione sufficiente per l'equilibrio spontaneo di un fluido sotto l'azione della propria gravità. Boll. Un. Mat. Ital. (3) **11** (1956), 499–503.

The mathematical content of this paper consists in a slight generalization of a theorem proved earlier by the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14** (1953), 779–783, p. 783]. The theorem concerns functions $y(x)$ that satisfy a differential equation of the form

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+f(y)=0,$$

and for which $y(x)>0$, $y'(x)<0$, in $0<x\leqq x^*$; $\lim_{x\to+0} y(x)=C<\infty$. Assuming $f(y)$ to be positive, non-decreasing, and sufficiently regular for $y\geqq 0$ it is proved that $y(x)$ decreases, in $0<x<x^*$, if $f(y)$ is increased. This theorem is applied to the problem of the stability of a fluid sphere under the action of its own gravity. It is shown that a stable equilibrium configuration is possible if the functional relation between pressure and density satisfies a certain simple inequality.

W. Wasow (Madison, Wis.).

See also: de Vito, p. 749; Lovass-Nagy, p. 780.

Partial Differential Equations

Heyn, Eugen. Über die eindeutige Bestimmtheit von Lösungen der Differentialgleichung $\Delta U=F(x, U)$ durch ihr Verhalten in einem Punkt. Math. Nachr. **15** (1956), 250–257.

The author starts with the following lemma: If $U(x)$ is a real function, non-identically zero, admitting continuous second derivatives within the p-dimensional sphere $|x|\leqq R_0$, and so that for each $n>0$, if $R\to 0$

$$\int_{\Omega_R}|u|^2ds=o(R)^n \text{ and} \int_{\Omega_R}|\Delta u|^2ds=o(R^n),$$

then for each $\varepsilon>0$, there can be determined a $R(\varepsilon)$, $0<R(\varepsilon)<R_0$, so that

$$\int_{\Omega_R}|\Delta u|^2ds>0 \text{ and} \int_{\Omega_R}|u|^2ds\leqq \varepsilon R^4\int_{\Omega_R}|\Delta u|^2ds.$$

Here Δ is the p-dimensional Laplace-operator and the integrals are taken over the surface of the p-dimensional sphere $|x|=R$.

From this lemma are deduced theorems where the uniqueness of the solution of the equation $\Delta U=F(x, U)$ is concluded from data on the behaviour of the solution U in a single point. As to the function F it is assumed, that for $|x-x_0|\leqq R_0$, $|U(x)-U_1(x)|\leqq\alpha$ there is a constant C so that

$$|F(x, U)-F(x, U_1)|\leqq C(x-x_0)^{-2}|u-u_1|.$$

In the principal theorem, from

$$\int_{|x-x_0|=R}|U_1-U_2|^2ds=o(R^n), \text{ for all } n>0,$$

it is deduced that $U_1(x)\equiv U_2(x)$ for $|x-x_0|\leqq R_0$.

The theorems are generalisations of results formerly published by T. Carleman [C. R. Acad. Sci. Paris **197** (1933), 471–474], and C. Müller [Comm. Pure Appl. Math. **7** (1954), 505–515; MR **16**, 42]. *H. Bremekamp.*

Mlak, W. The epidermic effect for partial differential inequalities of the first order. Ann. Polon. Math. **3** (1956), 157–164.

L'A. indica una condizione sufficiente affinché dalla $u(x^0, y_1, \cdots, y_n)\leqq v(x^0, y_1, \cdots, y_n)$ segua la

$$u(x, y_1, \cdots, y_n)\leqq v(x, y_1, \cdots, y_n)$$

in tutto un certo ipercubo contenuto nel semispazio $x\geqq x^0$ (e con una faccia contenuta nell'iperpiano $x=x^0$). Quivi la funzione $v(x, y_1, \cdots, y_n)$ è soluzione di un'equazione differenziale del tipo

$$\partial z/\partial x=f(x, y_1, \cdots, y_n, z, \partial z/\partial y_1, \cdots, \partial z/\partial y_n),$$

per la quale, in quel certo ipercubo, valga un teorema di unicità qualora si assegnino per le soluzioni i valori per $x=x^0$ e si ricerchino soluzioni continue insieme colle loro derivate prime. *G. Scorza-Dragoni* (Padova).

Pini, Bruno. Una generalizzazione del problema biarmonico fondamentale. Rend. Sem. Mat. Univ. Padova **25** (1956), 196–213.

Let D be a plane domain bounded by a single twice differentiable curve.

The author considers the biharmonic problem $\Delta\Delta u=0$ in D with a generalised boundary condition for the

unknown function. It permits to assign $L^{(1)}$-functions as data on the boundary, since the boundary condition is imposed to the unknown u by considering a certain limit, in the mean of the first order, on a continuous sequence of curves converging to the boundary.

The existence theorem for the classical problem is supposed to be known. Then an existence and uniqueness theorem is given for the generalised problem.

G. Fichera (Rome).

Èidel'man, S. D. Normal fundamental matrices of solutions of parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 523–526. (Russian)

Continuing his previous work [same Dokl. (N.S.) **98** (1954), 913–915; **103** (1955), 27–30; MR **17**, 857; 1092] the author considers a parabolic system

$$(*)\qquad \frac{\partial u}{\partial t}=A\left(t, x, \frac{\partial}{\partial x}\right)u$$

in the set $\{0\leqq t\leqq T, x\in D\}$ where D is now a bounded domain. A fundamental matrix solution of (*) is normal if the transposed matrix is a fundamental matrix solution of the system adjoint to (*). Sufficient conditions for the construction of a normal fundamental matrix $\omega(t, \tau, x, \xi)$ for (*) are given and estimates for the derivatives of $\omega(t, \tau, x, \xi)$ are obtained. By using the representation of a solution u of (*) in terms of $\omega(t, \tau, x, \xi)$, the author obtains differentiability conditions on u and conditions under which a solution u can be continued into the complex domain. Similar results are stated for an inhomogeneous parabolic system.

Finally, sufficient conditions are obtained in order that generalized solutions in the sense of S. L. Sobolev [The equations of mathematical physics, 3rd ed., Gostehizdat, Moscow, 1954; MR **16**, 1027] (apparently weak solutions) of (*), certain linear ellyptic systems, and certain systems involving higher derivatives with respect to t should be regular solutions. *J. Cronin.*

Volkovyskiĭ, L. I. On differentiability of quasi-conformal mappings. L'vov. Gos. Univ. Uč. Zap. **29**, Ser. Meh.-Mat. no. **6** (1954), 50–57. (Russian)

The author considers a class of quasi-conformal mappings of plane domains onto plane domains first introduced by Chabate [Mat. Sb. N.S. **17(59)** (1945), 193–210; MR **8**, 77] and establishes a sufficient condition under which such mappings are differentiable. This is achieved by an extension of a result due to E. Hopf [Math. Z. **34** (1931), 194–233]. *W. Seidel.*

Belinskiĭ, P. P. Behavior of a quasi-conformal mapping at an isolated singular point. L'vov. Gos. Univ. Uč. Zap. **29**, Ser. Meh.-Mat. no. **6** (1954), 58–70. (Russian)

The purpose of the paper is to establish the following result. Let $w=f(z)$ be a quasi-conformal mapping of $0<|z|\leqq 1$ onto a bounded domain in the w-plane and let

$$\iint_{0<|z|\leqq 1}[p(z)-1]|z|^{-2}d\sigma=A<\infty,$$

where $p(z)$ denotes the coefficient of dilatation of the mapping and $d\sigma$ the element of area in the z-plane. Then, $\lim_{z\to 0} w=w_0$ and $\lim_{z\to 0} (w-w_0)/z$ both exist and the second limit is different from 0 and ∞. For earlier results in this direction, cf. Teichmüller [Deutsche Math. **3** (1938), 621–678, p. 670] and Wittich [Math. Z. **51** (1948), 278–288; MR **10**, 241]. *W. Seidel* (Notre Dame, Ind.).

★**Vekua, I. N. Systeme von Differentialgleichungen erster Ordnung vom elliptischen Typus und Randwertaufgaben; mit einer Anwendung in der Theorie der Schalen.** Mathematische Forschungsberichte, II. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. 107 pp.

Authorized translation by Wolfgang Schmidt, with the consultation of the author, of the 1952 Russian edition reviewed in MR **15**, 230.

Žautykov, O. A. On the question of construction of integrals of first order partial differential equations with a countable number of independent variables. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 4(8), 48–69. (Russian)

The classical results on characteristics, first integrals, solution of Cauchy's problem, etc. are extended to equations

$$(\partial z/\partial t)+\sum_{k=1}^{\infty} a_k(t, x, z)(\partial z/\partial x_k)=a(t, x, z),$$

where $x=\{x_k\}$ $(k=1, 2, \cdots)$ is an element of a space X with norm $\|x\|=\sup |x_k|$. The coefficients a_k, a are assumed to satisfy the following conditions in a region $\alpha\leqq t\leqq\beta$, $\|x\|\leqq R$: a) they are continuous and uniformly bounded; b) they have continuous uniformly bounded partial derivatives with respect to each coordinate; c) if x', x'' are two points such that their first m coordinates coincide,

$$|a_k(t, x', z')-a_k(t, x'', z'')|\leqq\varepsilon_m(\|x'-x''\|+|z'-z''|),$$

where $\varepsilon_m\to 0$ when $m\to\infty$ (and a similar condition for a).

J. L. Massera (Montevideo).

★**Будак, Б. М.; Самарский, А. А.; и Тихонов, А. Н. [Budak, B. M.; Samarskiĭ, A. A.; and Tihonov, A. N.] Сборник задач по математической физике. [Collection of problems in mathematical physics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 685 pp. 14.45 rubles.

The present book is intended to supplement the usual text on partial differential equations of physics so as to enable a student (or a graduate enginer faced with a theoretical problem) to acquire the facility to formulate and solve boundary value problems of classical physics.

The first part of the book is devoted to the listing of a large number of problems. The chapter divisions are made according to the type of differential equation considered, i.e., hyperbolic, parabolic, or elliptic. A brief statement is given in each chapter as to the branches of classical physics where these particular equations are likely to arise. The second part of the book gives the answers together with the methods needed to find them. The treatment of the first problems of a given classification starts with the selection of the variables and the formulation of the problem as a partial differential equation. The methods of solution discussed primarily are the separation of variables and integral representations. The amount of detail given appears to be adequate, typical problems being worked out step by step. After going through a detailed example, a student should find sufficient the indications on procedure given for similar problems.

In view of the many advantages of this book, it is unfortunate that it is open to serious objections on two counts: 1. The division into equation types is rather artificial and necessitates the repetition of solution

methods, e.g., separation of variables, in each chapter. If this duplication were avoided, considerable additional material, such as operational or variational methods, could be included without expanding the book. 2. Though supposedly intended for the practicing engineer as well as for a student taking a course, the book does not lend itself readily to occasional reference use. Considerable material may have to be studied before finding out how to attack a given problem. The lack of an index increases the difficulty. *J. E. Rosenthal* (Passaic, N.J.).

Mizel, Victor J. A boundary layer problem for an elliptic equation in the neighborhood of a singular point. Proc. Amer. Math. Soc. 8 (1957), 62–67.

In Ann. of Math. (2) **51** (1950), 428–445 [MR 11, 439], N. Levinson studied the asymptotic behavior, as $\varepsilon\to 0$, of the solution of

$$(1)\qquad \varepsilon\Delta u+A(x,y)u_x+B(x,y)u_y-k^2(x,y)u=D(x,y)$$

with prescribed values on the boundary S of a bounded domain R. He assumed that the system

$$(2)\qquad dx/dt=-A(x,y),\ dy/dt=-B(x,y)$$

had no singularities in $R+S$. In the present paper the system (2) is permitted to possess a finite number of stable nodes or foci in $R+S$, and Levinson's results are extended to this case. It is shown that, if the data of the problem are in $C^{(6)}$, the solution of (1) tends, as $\varepsilon\to 0$, to a limit everywhere in R except possibly at the singularities of (2) and along those solution curves of (2) which are tangent to S. This limit function of u solves the reduced equation obtained from (1) by setting $\varepsilon=0$ and approaches the prescribed boundary conditions at those arcs of S where the vector $(A(x,y),B(x,y))$ points outward. The proof resembles much the one in Levinson's paper to which the reader is frequently referred.

W. Wasow (Madison, Wis.).

Berg, Paul W. On univalent mappings by solutions of linear elliptic partial differential equations. Trans. Amer. Math. Soc. **84** (1957), 310–318.

Generalizing results of H. Lewy and E. Heinz, the author proves the following theorem about two solutions u, v of a pair of second order linear elliptic partial differential equations with identical principal parts and with coefficients in suitable Hölder classes. If u, v generate a homeomorphism of the unit circle onto itself and vanish at the origin, then $u_x^2+u_y^2+v_x^2+v_y^2\geq\mu$ there, where μ is a positive constant depending only on the coefficients of the differential equations. The proof is based on the similarity principle of Bers, on some a priori estimates of Schauder, on an integral inequality, and on geometrically motivated arguments about the local behaviour of the mapping. *P. R. Garabedian* (Stanford, Calif.).

Bergman, Stefan. Multivalued harmonic functions in three variables. Comm. Pure Appl. Math. **9** (1956), 327–338.

The Bergman integral operator for Laplace's equation in three dimensions maps ordered pairs of analytic functions, $\chi_1(\tau,\tau^*)$, $\chi_2(\tau,\tau^*)$, of two complex variables into solutions of

$$(\partial^2\tilde{H})/(\partial x)^2+(\partial^2\tilde{H})/(\partial y)^2+(\partial^2\tilde{H})/(\partial z)^2\equiv (\partial^2 H)/(\partial x)^2-(\partial^2 H)/(\partial Z\partial Z^*)=0,$$

and may be defined in three steps:

$$\chi(\tau,\tau^*)=2(\tau\tau^*)^{\frac12}\chi_1(\tau,\tau^*)+\chi_2(\tau,\tau^*),$$

$$f(u,\zeta)\equiv 2\int_0^1 u^{\frac12}\{d[u^{\frac12}\chi(u\zeta^{-1}T^2,\,u\zeta(1-T)^2)]/du\}dt,$$

$$C_3(\chi)\equiv B_3(f,L;\vec{X}_0)=(2\pi i)^{-1}\int_L f(u,\zeta)\zeta^{-1}d\zeta.$$

Here (x,y,z) belongs to some neighborhood of the point $\vec{X}_0=(x_0,y_0,z_0)$, and L is some rectifiable (not necessarily closed) curve in the ζ-plane.

If L is the circle $|\zeta|=1$, and $\vec{X}_0=(0,0,0)$, the inverse mapping is given by:

$$\chi(Z,Z^*)=H(2(ZZ^*)^{\frac12},Z,Z^*).$$

As a consequence of their characteristic property of mapping analytic functions into solutions of differential equations, integral operators permit the use of function theory in the study of solutions. In the present paper, Weierstrass's representation of algebraic functions on a Riemann surface is translated into a form of representation for a certain class of multi-valued harmonic functions in three dimensions. Extensions to other equations and to systems of equations are discussed. *R. B. Davis.*

Pettineo, Benedetto. Sulla funzione di Green pel problema di Dirichlet relativo alle equazioni lineari ellittiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 306–311.

Valendosi di un metodo di M. Picone [same Rend. (8) **2** (1947), 365–371, 485–492, 717–725; MR **9**, 145, 286, 287] per la traduzione in un sistema di equazioni di Fischer-Riesz dei problemi al contorno relativi alle equazioni ellittiche, l'A. perviene a determinare formalmente uno sviluppo in serie per la funzione di Green del problema di Dirichlet relativo a un'equazione ellittica in m variabili. La convergenza (in media) di tale serie è dimostrata solo nel caso di due o tre variabili. *C. Miranda.*

Minasyan, R. S. On the solution of the Dirichlet problem over a rectangle for equations with non-separable variables. Akad. Nauk Armyan. SSR. Dokl. **23** (1956), 145–152. (Russian. Armenian summary)

A Fourier series solution is given for elliptic equations of the form $U_{xx}+2\alpha U_{xy}+\beta U_{yy}=P$ $(\beta-\alpha^2>0)$ over a rectangle, it being assumed that P is summable on the reactangle and that the boundary functions are continuous and have summable first derivatives.

M. G. Arsove (Seattle, Wash.).

Szmydt, Z. Sur une généralisation des problèmes classiques concernant un système d'équations différentielles hyperboliques du second ordre à deux variables indépendantes. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 579–584.

Let $u=(u^1,\cdots,u^n)f$, g, h be n-vectors; x, y scalars; $y=\alpha_k(x)$, $x=\beta_k(y)$ continuous functions for $|x|\leq a$, $|y|\leq b$, $k=1,\cdots,n$, satisfying $|\alpha_k|\leq b$, $|\beta_k|\leq a$, respectively. The author states that, by successive approximations, one can prove existence and uniqueness for a solution of the following problem: $u_{xy}=f(x,y,u,u_x,y_y)$, $u(x_0,y_0)=u_0$, $u_x^k=g^k(x,u,u_y)$ for $y=\alpha_k(x)$, $u_y^k=h^k(x,u,u_x)$ for $x=\beta_k(y)$ if it is assumed that f, g, h are continuous for $|x|\leq a$, $|y|\leq b$ and $|u|$, $|u_x|$, $|u_y|\leq c$, are bounded (with certain restrictions on their bounds), and satisfy uniform Lipschitz conditions with respect to the variables u, u_x, u_y with sufficiently small

Lipschitz constants. As in Hartman and Wintner [Amer. J. Math. 74 (1952), 834–864; MR 14, 475], if the Lipschitz condition with respect to u is omitted, the existence (but not uniqueness) can be proved. The author outlines a proof depending on Schauder's fixed-point theorem.

P. Hartman (Baltimore, Md.).

Gagliardo, Emilio. Problema al contorno per equazioni differenziali lineari di tipo parabolico in n variabili. Ricerche Mat. 5 (1956), 169–205.

Let E be a bounded domain in the space $x=(x_1, \cdots, x_m)$ with a boundary $\mathscr{F}E$ that can be divided into a finite number of sets $F_1, \cdots, F_h$ such that each F_k has a parametric representation $x_j=f_j(\xi_1, \cdots, \xi_{m-1})$, $j=1, \cdots, m$, with the f_j belonging to C^2. Use D to denote the cylinder $D\equiv(y_0\leqq y\leqq y_1;\ x\in E)$. Let B_0 stand for the base $y=y_0$ of D, and let L denote the hypersurface $L\equiv(y_0\leqq y\leqq y_1;\ x\in \mathscr{F}E)$. A function $v=v(y, x)$ defined in D will be said to belong to class μ if it is in each of its variables separately absolutely continuous, has (almost everywhere) the derivativates u_{x_i} which are absolutely continuous in each of the variables $x_1, \cdots, x_m$, and the derivatives u_{x_i}, u_y, $u_{x_ix_j}$ are square summable in D. Assume there are positiev constants α_0, α_1 such that

$$\alpha_0 \sum_j^{1\cdots m} \lambda_j^2 \leqq \sum_{ij}^{1\cdots m} a_{ij}(y, x)\lambda_i\lambda_j \leqq \alpha_1 \sum_j^{1\cdots m} \lambda_j^2,$$

and consider the first boundary value problem for the parabolic equation

$$(*)\quad \sum_{ij}^{1\cdots m} a_{ij}(y, x)u_{x_ix_j}-u_y+\sum_j^{1\cdots m} a_j(y, x)u_{x_j}+a(y, x)u=f(y, x),$$

$$(**)\qquad u=\psi \text{ on } B_0,\ u=\varphi \text{ on } L\ \ (\psi=\varphi \text{ on } B_0\cap L),$$

where the a_{ij} are continuous in D, the a_i, a are bounded and measurable there, and f is square summable on D. The assigned boundary functions $\psi=\psi(x)$ and $\varphi=\varphi(y, \xi_1, \cdots, \xi_{m-1})$ are assumed to be absolutely continuous with respect to each of their arguments (separately), and the functions ψ_{x_i}, φ_y, φ_{ξ_i}, $\varphi_{\xi_i\xi_j}$ are assumed to be square summable. In the present paper the author proves the existence of a function $u=u(y, x)$ belonging to class μ which solves the above mentioned boundary value problem in the following generalized sense: u in D satisfies (*) almost everywhere and assumes on B_0+L almost everywhere the assigned values.

F. G. Dressel (Durham, N.C.).

Gagliardo, Emilio. Teoremi di esistenza e di unicità per problemi al contorno relativi ad equazioni paraboliche lineari e quasi lineari in n variabili. Ricerche Mat. 5 (1956), 239–257.

In this paper it is proved that the boundary value problem stated in the preceding review has a unique solution in the class μ if the function $\psi=\psi(x)$ assigned on B_0 is assumed to have the additional properties that its second derivatives ψ_{xx} exist and are square summable on B_0. Similar results (existence and uniqueness of generalized solutions in class μ) are proved relative to the first boundary value problem for the quasi-linear parabolic equation

$$\sum_{ij}^{1\cdots m} a_{ij}(y, x)u_{x_ix_j}-u_y=F(y, x, u, u_{x_1}, \cdots, u_{x_m}),$$

where F has the form

$$F(y, x, u, p_1, \cdots, p_m)=\sum_1^{1\cdots m} a_i(y, x, u, p_1, \cdots, p_m)p_i+ a(y, x, u, p_1, \cdots, p_m)u+f(y, x, u, p_1, \cdots, p_m).$$

In the preceding review the assumption on the $a_{ij}(y, x)$ have been stated. In the variables y, x the functions a_i, a, f are assumed to be measurable and are assumed to be continuous as functions of their other arguments. The a_i, a are bounded, and further it is assumed that

$$|f(y, x, u, p_1, \cdots, p_m)|\leqq g(y, x),$$

where g is a function which is square summable on D.

F. G. Dressel (Durham, N.C.).

Ciliberto, Carlo. Nuovi contributi alla teoria dei problemi al contorno per le equazioni paraboliche non lineari in due variabili. Ricerche Mat. 5 (1956), 206–225.

The questions of existence and uniqueness of a solution $v=v(x, y)$ in $T\equiv[0\leqq x\leqq X, 0\leqq y\leqq Y]$ of the parabolic equation

$$(*)\qquad F(x, y, v, v_x, v_y, v_{xx})=0\ \ (F_{v_y}F_{v_{xx}}<0)$$

taking on the boundary value $v(x, 0)=v(0, y)=v(X, y)=0$ are treated in the present paper by topological methods. Assume there exists a one parameter family of parabolic differential equations

$$(**)\qquad G(x, y, v, v_x, v_y, v_{xx}, \alpha)=0$$

which for the parameter value $\alpha=1$ reduces to equation (*), and such that the function G belongs to class C^3 with respect to all its arguments, and further that

$$G=G_x=G_{xx}=G_y=0$$

for $x=0$ or $x=X$ when $y=v=v_x=v_y=v_{xx}=0$, $0\leqq\alpha\leqq1$. Under these conditions the above mentioned boundary value problem for equation (*) is shown to possess a unique solution if the following properties hold relative to solutions of the corresponding boundary value problem for the family (**) of parabolic equations: (i) There exists a unique solution for some value of the parameter; (ii) as α varies, all possible solutions are uniformly bounded. (In an earlier paper [Richerche Mat. 3 (1954), 129–165; MR 16, 1028] the author treated this problem under somewhat weaker continuity conditions on G but with the additional assumption that as α varies any resulting solutions were equicontinuous as well as uniformly bounded.) In the last part of the paper similar results, but with fewer restrictions, are obtained for the special forms

$$v_{xx}=f(x, y, v, v_x, v_y),\ \ v_y=g(x, y, v, v_x, v_{xx})$$

of the equation (*).

F. G. Dressel (Durham, N.C.).

Pogorzelski, W. Problème aux limites pour l'équation parabolique dont les coëfficients dépendent de la fonction inconnue. Ricerche Mat. 5 (1956), 258–272.

Use $A\equiv(x_1, \cdots, x_n)$ to denote a point of the domain Ω, and $P\equiv(x_1, \cdots, x_n)$ to denote a point of the boundary S of Ω. The transversal derivative du/dT_P associated with the parabolic equation

$$(*)\quad \sum_{i,j=1}^{n} a_{ij}(A, t, u)u_{x_ix_j}+\sum_{j=1}^{n} b_j(A, t, u)u_{x_j}+c(A, t)u-u_t= \lambda F(A, t, u)$$

is defined as follows

$$(**)\ \frac{du}{dT_P}= \lim_{A\to P}\sum_{i,j=1}^{n} a_{ij}(A, t, u(A, t))\cos(N_P, x_j)\frac{\partial u\,(A, t)}{\partial x_i}.$$

Here N_P denotes the normal to the surface S at the point P. Assume the quadratic form $\sum a_{ij}(A, t, u)X_iX_j$ is positive definite for $A \in \Omega+S$, $0 \leq t \leq T$, $|u| \leq R$; and that also on this set each of the $a_{ij}(A, t, u)$ satisfies a Hölder condition with respect to the arguments A, t, u. The functions $b_j(A, t, u)$, $c(A, t)$, and $F(A, t, u)$ are assumed to be continuous and in the arguments A, u to satisfy Hölder conditions. Consider the problem of finding a solution $u=u(A, t)$ of equation (*) for $A \in \Omega$, $0<t<T$, and such that

$$\frac{du}{dT_P}=vG(P, t, u(P, t)), \ \lim_{t\to 0} u(A, t)=0,$$

where $G(P, t, u)$ is a preassigned function. If the surface S satisfies certain regularity conditions, and if the auxiliary parameters λ, v are sufficiently small the author, using topological methods [J. Schauder, Studia Math. 2 (1930), 171–180], proves that this boundary value problem has at least one solution. *F. G. Dressel* (Durham, N.C.).

Bicadze, A. V. On a problem of Frankl'. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 1091–1094. (Russian)

A boundary value problem for an equation of mixed type recently formulated by F. I. Frankl' [Prikl. Mat. Meh. **20** (1956), 196–202; MR **18**, 255] has led the author to consider the following problem for

(*) $$U_{xx}+(\operatorname{sgn} y)U_{yy}=0.$$

Let A be $(0, 1)$, $A'(0, -1)$, $C(1, 0)$, and $B(a, 0)$ $(a \geqq 1)$. Let σ be a smooth curve from B to A such that for its parametric representation $x(s)$, $y(s)$ in terms of arc length s from C $dy/ds \geqq 0$. Let D be the interior of the contour composed of σ and the segments AA', $A'C$, and CB, and let D_1 be the part of D in $y>0$. In D find a solution of (*) such that on σ $U=\psi_1$; on CB $U=\psi_2$; and on AA' $U_x=0$ and $U(0, y)-U(0, -y)=f(y)$, where ψ_1, ψ_2, and f satisfy Hölder conditions. The author finds a unique solution continuous in the closure $\bar{D}$ of D, regular except on $y=0$ and $y=-x$, and with U_x and U_y continuous except possibly on these lines and at the corners of the boundary of D. The problem is reduced to finding in D_1 an analytic function $U+iV$ of $z=x+iy$ such that $V(0, y)=0$; $U=0$ on σ and CB; and $U(x, 0)+V(x, 0)=U(0, x)+f(-x)$ for $0 \leq x \leq 1$. If $\xi+i\eta=\zeta=\omega(z)$ maps D_1 conformally onto the first quadrant of the unit circle in the ξ-plane, then $U(0, \eta)$ is defined as a solution of a linear singular integral equation, and then $U+iV=\Phi(\xi)$ can be found by a quadrature. *J. Giese* (Aberdeen, Md.).

Fourès-Bruhat, Y. Solution élémentaire d'équations ultra-hyperboliques. J. Math. Pures Appl. (9) **35** (1956), 277–288.

Construction d'une solution élémentaire de l'opérateur ultra-hyperbolique

$$\square=\sum_{i=1}^{n_1}\frac{\partial^2}{(\partial x^i)^2}-\sum_{j=1}^{n_2}\frac{\partial^2}{(\partial x^j)^2}.$$

Si n_1 et n_2 sont impairs, l'auteur détermine cette solution élémentaire sous la forme d'une distribution E satisfaisant à l'équation $\square E=\delta$ dans $R^{n_1+n_2}$ (δ: fonction de Dirac) et constituée par une somme finie de dérivées transversales de distributions portées par le cône caractéristique Σ de l'opérateur $\square$, d'équation

$$\sum_{i=1}^{n_1}(x^i)^2-\sum_{j=1}^{n_2}(x^j)^2=0.$$

Ces distributions sont en fait de simples puissances d'un paramètre variant sur une génératrice de Σ et sont obtenues comme solutions d'un système d'équations différentielles ordinaires. L'auteur passe aux autres valeurs de n_1 et n_2 par la méthode de descente. *H. G. Garnir.*

Marhasev, G. On a boundary problem for the equation $\frac{\partial}{\partial x}\Delta u=0$. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 926–928. (Russian)

It was shown by Hadamard [Enseignement Math. **35** (1956), 5–42] that every solution of the equation

1) $$\frac{\partial}{\partial x}\Delta u=0$$

is of the form $u=v+w$, where v is harmonic and w satisfies $\partial w/\partial x=0$. The author seeks solutions of 1) in $K: x^2+y^2<1$ subject to the boundary conditions $u|_{x^2+y^2=1}=f_1(\vartheta)$ and $u|_{x=0}=f_2(y)$, where f_1 and f_2 are given. If $\mu(\vartheta)=v|_{x^2+y^2=1}$, the equation 1) is equivalent to

2) $$\mu(\vartheta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mu(t)\frac{1-\rho^2_\vartheta}{1+\rho^2_\vartheta-2\rho_\vartheta \sin t}dt+F(\vartheta),$$

where $\rho_\vartheta=\sin\vartheta$, $F(\vartheta)=f_1(\vartheta)=f_2(\sin\vartheta)$. Asserting that the known proofs of existence and uniqueness of solutions of 2) are long and complicated, the author constructs a unique solution of 2) in explicit form.

A. J. Lohwater (Ann Arbor, Mich.).

Borok, V. M. The solution of Cauchy's problem for certain types of systems of linear partial differential equations. Amer. Math. Soc. Transl. (2) **5** (1957), 285–304.

A translation of the 1955 Russian paper reviewed in MR **16**, 929.

Kostyučenko, A. G.; and Šilov, G. E. On the solution of Cauchy's problem for regular systems of linear partial differential equations. Amer. Math. Soc. Transl. (2) **5** (1957), 275–283.

A translation of the authors' 1954 Russian paper reviewed in MR **16**, 253.

Pliś, A. Generalisation of the Cauchy problem for a system of partial differential equations. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 741–744.

The system of partial differential equations considered is of the form

$$u^i{}_x=f^i(x, Y, u^1, \cdots, u^m, u^i{}_{y_1}, \cdots, u^i{}_{y_n}) \ (i=1, \cdots, m),$$

where $Y=(y_1, \cdots, y_n)$. The generalization of Cauchy's problem treated is that of finding a solution $u^i(x, Y)$ of this system for which

$$u^i(a_i, Y)=\omega^i(Y) \ (i=1, \cdots, m),$$

where the a_i denote constants and the ω^i are prescribed functions. If the f^i are Lipschitz-continuous in their last $m+n$ arguments, and if the $|a_i|$ are sufficiently small, it is proved that any solution having a complete differential is unique. If the f^i are Lipschitz-continuous in all but their first argument, if the ω^i too are Lipschitz-continuous, and if the $|a_i|$ again are sufficiently small, it is proved that a C^1 solution exists. *A. Douglis.*

Mergelyan, S. N. Harmonic approximation and approximate solution of the Cauchy problem for the Laplace equation. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 3–26. (Russian)

Let σ be a sufficiently smooth homoeomorph of a disc

embedded in 3-dimensional Euclidean space. The author considers the problem of approximating on σ simultaneously a continuous function f_1 by a harmonic polynomial and a second continuous function f_2 by the normal derivative of this polynomial. An explicit bound for the degree of approximation is found which involves the degree of the approximating polynomial, the modulus of continuity of f_1 and f_2 and a function measuring the deviation of σ from a flat surface. [Theorem 1 of the paper states this bound incorrectly, since it does not give the right answer for a flat disc. The proof seems correct except for the assumption that (14) defines a number a tending to zero with h.] If σ is replaced by the homeomorph τ of a sphere, then a necessary condition for simultaneous approximability to within ε of f_1 by H, f_2 by $\partial H/\partial n$ is

$$\iint_\tau \Big(f_1(Q)\frac{\partial}{\partial n}\frac{1}{r_{PQ}}-\frac{1}{r_{PQ}}f_2(Q)\Big)dS=0$$

for P outside τ. For sufficiently smooth τ this condition is also sufficient.

Weighted approximation on σ is also considered and proved possible (within ε), if the weight function tends to zero like $\exp(-|QQ_0|^{-p})$ ($p>2$) near a point $Q_0 \in \sigma$. The author also treats several problems of majorisation for harmonic functions. An example is the following result. If $u(P)$ is harmonic in the 3-dimensional unit-sphere and continuous together with its first partial derivatives on the boundary of the sphere, if further for a fixed point A on the boundary and a variable point P

$$|u(P)|+\left|\frac{\partial u}{\partial n}(P)\right|<\exp(-|PA|^{-p}),$$

$p>2$, then $u\equiv 0$. *W. H. J. Fuchs.*

See also: Weinstein, p. 729; Duff, p. 730; Bellman, p. 744; de Vito, p. 749; Ritchie and Sakakura, p. 780.

Difference Equations, Functional Equations

Milošević, Kovina. Décomposition d'une différence finie d'une fonction suivant les différences de ses dérivées. Bull. Soc. Math. Phys. Macédoine 6 (1955), 5–8. (Serbo-Croatian. French summary)

Utilizing his own interpolation formula, Mikeladze [Dokl. Akad. Nauk SSSR (N.S.) 92 (1953), 479–482; MR 15, 609] has shown that the finite difference of the function can be expressed by means of the differences of their derivatives

$$\Delta^n f(a+\lambda h)=h^n\sum_{\rho=0}^{r}A_{n\rho}\Delta^\rho f^{(n)}(a)+R_{nr}\quad(\lambda\leqq 0).$$

The coefficients $A_{n\rho}$ of the series can be obtained by aid of the integral relations. In this paper the author shows that the coefficients $A_{n\rho}$ can be expressed as the Stirling's numbers of the 1st and 2nd kinds [Jordan, Calculus of finite differences, Budapest, 1939]. In the special case, when $\lambda=0$, the above formula reduces to the known formula of the calculus of finite differences:

$$A_{n\rho}=\frac{n!}{\rho!}\sum_{m=0}^{\rho}\frac{m!}{(m+n)!}\,S_\rho{}^m\sigma_{m+n}{}^n.$$

D. Rašković (Belgrade).

Bajraktarević, Mahmud. Sur certaines solutions de deux équations fonctionnelles. Bull. Soc. Math. Phys. Serbie 6 (1954), 172–184. (Serbo-Croatian. French summary)

In the class of functions $\varphi(t)$, $M(t)$, $w(t)$ that have second derivatives on an interval I: $a\leqq t\leqq b$, and are such that $a(t)\neq 0$ and $\varphi(t)$ and $M(t)$ are monotone functions with inverses φ^{-1} and M^{-1}, respectively, the author studies the equation

$$\varphi^{-1}\Big(\sum_{i=1}^{n}\varphi_i/n\Big)=M^{-1}\Big(\sum_{i=1}^{n}M_iw_i\Big/\sum_{i=1}^{n}w_i\Big),$$

where values t_i ($i=1,\cdots,n$; $n>2$) are on I and Q_i denotes $Q(t_i)$ for any function $Q(t)$. It is shown that the totality of solutions is given by $M=\alpha+\beta/w$, $\varphi=\gamma w+\delta$, where $w=w(t)$ is an arbitrary function and α, β, γ, δ are arbitrary constants except that $\beta\gamma w\neq 0$.

An analogous integral equation also is studied; both necessary and sufficient conditions are discussed; and some solutions, but not necessarily all solutions, are obtained. *E. F. Beckenbach* (Los Angeles, Calif.).

Bellman, Richard. Functional equations in the theory of dynamic programming. VI. A direct convergence proof. Ann. of Math. (2) 65 (1957), 215–223.

The following theorem is proved: If for $-c_1\leqq x\leqq c_1$, $-\infty<m_1\leqq y\leqq m_2<\infty$, $F(x,y)$ and $G(x,y)$ are continuous and satisfy with respect to x a Lipschitz-condition of exponent $0<a\leqq 1$, $a^2+a>1$, further $|g(x,y)|\leqq a_1|x|+b_1$ then the subsequence $\{f_{2^n}\}$ of the sequence of functions

$$f_0(c)=\max_{y(0)}\frac{T}{n}F[c,y(0)],$$

$$f_{k+1}(c)=\max_{y(0)}\Big(\frac{T}{n}F[c,y(0)]+f_k[c+\frac{T}{n}G(c,y(0))]\Big)$$
$$(c=x(0),\ R_i[c,y(0)]\leqq 0,\ i=1,2,\cdots,k)$$

converges uniformly in an interval $[-c_2,c_2]\in[-c_1,c_1]$ for sufficiently small T. Relations of the problem with multidimensional eigenvalue and variational problems and multi-stage games are mentioned and will be treated in other papers of this series. *J. Aczél* (Debrecen).

Gołąb, S. Zum distributiven Gesetz der reellen Zahlen. Studia Math. 15 (1956), 353–358.

The author has asked for the most simple conditions for the functions f and g in $g[f(x,y),z]=f[g(x,z),g(y,z)]$ that are sufficient for the automorphism of f and g with respect to addition and multiplication in the field of real numbers.

He sent his solution to J. Łoś in 1953 and published it in Rocznik Naukowo-Dydaktyczny W. S. P. w Krakowie 1 (1954), 3–10. M. Hosszú has given a solution under special regularity conditions for f and g [Acta Math. Acad. Sci. Hungar 4 (1953), 159–167; MR 15, 324]. Because the paper of 1954 is not readily available and because he has simplified one of the original conditions, the author has published the result here once more. *J. A. Schouten* (Epe).

See also: Polniakowski, p. 732.

Integral and Integrodifferential Equations

See: Bicadze, p. 743; Bajraktarević, p. 744; Riesz and Sz.-Nagy, p. 747; Crease, p. 776.

Calculus of Variations

De Sloovere, H. Le calcul des variations successives d'une intégrale multiple, par la méthode invariantive de Th. De Donder. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 1028–1032.

This paper completes the calculation of the second

variation of a multiple integral, as treated in earlier work by the author [same Bull. (5) 39 (1953), 948–952; MR 15, 634]. *J. L. Synge* (Dublin).

Bourgin, D. G. Arrays of compact pairs. Ann. Mat. Pura Appl. (4) 40 (1955), 211–221.

This is a precise, compressed, but quite readable modern account of the Morse theory. We quote the author's own summary: "The properties of direct limits of Čech homology groups for a non-cofinal collection of compact pairs are explicitly stated with a viewto wards applications. The usual axioms including full excision are satisfied under obvious restrictions. Some of the Morse rank and span concepts are taken up in terms of these limit groups with the slight change of homology classes rather than cycles. A simple demonstration using the intuitive notions of the various cap types is presented for two basic exact sequences recently given by Deheuvels." *R. Bott* (Princeton, N.J.).

Singbal, B. V. Algebraic basis of Morse's variational theory. J. Indian Math. Soc. (N.S.) 18 (1954), 131–165 (1955).

This is another modern exposition of the rank and span theory of M. Morse. Influenced by notes of Deheuvels [C. R. Acad. Sci. Paris 235 (1952), 778–780, 858–860, 1270–1272; MR 14, 492], it parallels his theory. As the author points out, the main difference is that his definition of critical levels uses direct, rather than inverse, limits. *R. Bott* (Princeton, N.J.).

See also: Zoltán, p. 782.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Montgomery, Deane. Finite dimensionality of certain transformation groups. Illinois J. Math. 1 (1957), 28–35.

The main theorem of the paper is as follows: If G is a locally compact effective transformation group of a manifold, then G is finite-dimensional. Since an infinite-dimensional locally compact group contains an infinite-dimensional compact subgroup, G may be assumed to be compact. The main theorem can be then easily obtained from the following: Let G be a compact connected group acting on an n-dimensional manifold M and let F be the set of points of M left fixed by every transformation of G. If dim $F \geqq n-1$, then $F=M$.

An outline of the proof of the latter theorem is as follows. Using the fact that G is an inverse limit of compact connected Lie groups, the proof is first reduced to the case where G is a solenoid. Suppose, therefore, that G is a solenoid and denote by ϕ the natural map $M \to M/G$. Assume now that the assertion of the theorem is false, i.e. that $F \neq M$, and take a point p on the boundary B of $M-F$ such that any sufficiently small neighborhood of p is separated by B. A contradiction is then deduced by showing that for a suitably chosen invariant compact set X in a neighborhood of p and for the boundary A of X, $H_n(\phi(X), \phi(A)) \neq 0$ on one hand and $H_n(\phi(X), \phi(A))=0$ on the other. *K. Iwasawa* (Cambridge, Mass.).

Ellis, Robert. A note on the continuity of the inverse. Proc. Amer. Math. Soc. 8 (1957), 372–373.

Theorem: If G is a locally compact Hausdorff space which is endowed with a continuous associative multiplication under which it is a group, then inversion is continuous. *A. D. Wallace* (New Orleans, La.).

Braconnier, Jean. Remarques sur les groupes localement compacts dont les structures uniformes droite et gauche sont égales. Ann. Univ. Lyon. Sect. A. (3) 18 (1955), 15–19.

L'auteur annonce le théorème (évidemment incorrect) que tout groupe localement compact, dont les structures uniformes droite et gauche sont égales, est produit direct d'un sous-groupe compact et d'un sous-groupe euclidien. Le théorème correct et démontré se rapporte à des groupes qui satisfont à la condition supplémentaire d'être connexes. Ce théorème a été démontré pour la cas séparable par H. Freudenthal [Ann. of Math. (2) 37 (1936), 57–77]. L'auteur de la note présente constate une lacune dans cette démonstration. Cette remarque est formellement correcte, mais la lacune n'est qu'apparente. Pour la combler, il suffit d'admettre qu'au no. 35 du mémoire cité l'élémente a est arbitraire (au lieu d'être élément de W).

H. Freudenthal (Utrecht).

See also: Goetz, Hartman and Steinhaus, p. 724.

Lie Groups and Algebras

Ehrenpreis, L.; and Mautner, F. I. Some properties of the Fourier transform on semi-simple Lie groups. II. Trans. Amer. Math. Soc. 84 (1957), 1–55.

[Part I appeared in Ann. of Math. (2) 61 (1955), 406–439; MR 16, 1017.] The authors now study functions $f(g)$ on the group G of all conformal mappings of the interior of the unit circle. They introduce the class of spherical functions of type m, n: these are functions $f(g)$ which satisfy $f(k_\theta^{-1}gk_\phi^{-1})=e^{2\pi in\theta}f(g)e^{2\pi im\phi}$ for all g and all rotations k_θ, k_ϕ through "angles" θ, ϕ. They study the structure of various group-algebras on G, particularly the algebra D of all infinitely differentiable functions of compact support, with convolution as product. First they study the subspace $D_{m,n}$ consisting of all spherical functions of type m, n: the Fourier transform of $D_{m,n}$ consists of all numerically-valued entire functions of exponential type satisfying certain conditions. By using this result they characterize the Fourier transform of D by proving an analogue of the Paley-Wiener theorem, so that D turns out to be isomorphic to a matrix algebra where the matrix coefficients are certain entire functions of exponential type. The authors also introduce the space E of all infinitely differentiable functions with the Schwartz topology. The dual space E' of distributions is studied and again characterized as a matrix algebra whose matrix coefficients are entire functions of exponential type. Schwartz's theory of mean periodic functions is carried over to spherical functions of type m, n. Finally the authors introduce a more general notion of mean periodic functions: if $f(g) \in E$, it is called two-sided mean periodic if the closure in E of the set $D*f*D$ is not E; they obtain an expansion theorem for these functions.

R. P. Boas, Jr. (Evanston, Ill.).

Stoka, Marius I. Sur les sous-groupes d'un groupe G_r mesurable. Com. Acad. R. P. Romîne 6 (1956), 393–394. (Romanian. Russian and French summaries)

Freudenthal, Hans. Zur Berechnung der Charaktere der halbeinfachen Lieschen Gruppen. III. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 511–514.

Given a Lie group G and a closed subgroup F we can form the quotient space G/F. If we can find a linear representation of G and a vector v in the representation space V such that $F=\{\sigma|\sigma\in G,\ \sigma v=v\}$, then G/F can be identified with the orbit of v under the action of G. Otherwise put, G/F can be embedded in V in such a way that the natural action of G on G/F can be extended to be a linear representation of G on V. This paper makes explicit calculations of a representation of $\mathfrak{E}_8$ such that a subgroup isomorphic to $\mathfrak{D}_8$ is the isotropy group of a certain vector, thereby embedding $\mathfrak{E}_8/\mathfrak{D}_8$ in a space of dimension 27000. A number of corrections to parts I and II [same Proc. 57 (1954), 369–376, 487–491; MR 16, 673] appear in a footnote. *A. M. Gleason.*

See also: Herstein, p. 714; Merriell, p. 715; Hermann, p. 762.

Topological Vector Spaces

Klee, V. L., Jr. An example in the theory of topological linear spaces. Arch. Math. 7 (1956), 362–366.

Consider the following properties of a linear topological space E: (a) If $0\neq x\in E$ there exists f in E^*, the set of continuous linear functionals on E, such that $f(x)\neq 0$. (b) 0 is the only continuous linear functional on E. An example is given of a metric linear space E with supplementary closed subspaces M and N such that E satisfies (a), but E/M and E/N both satisfy (b). *M. M. Day.*

Dieudonné, Jean. Denumerability conditions in locally convex vector spaces. Proc. Amer. Math. Soc. 8 (1957), 367–372.

Let $\mathscr{B}$ be a set of bounded subsets of a locally convex topological vector space E. By a fundamental system of sets of $\mathscr{B}$ the author means a subset $\mathscr{F}$ of $\mathscr{B}$ such that every member of $\mathscr{B}$ is a subset of a member of $\mathscr{F}$. For each choice of $\mathscr{B}$ one can impose a condition on E by demanding that $\mathscr{B}$ should admit a denumerable fundamental system. In this note the author investigates some of the implications of such conditions. When $\mathscr{B}$ is the set of all convex compact sets and E is a t space then the denumerability condition in question implies that E is the strong dual of a space which is both a Montel space and a Frechet space. When $\mathscr{B}$ is the set of all compact sets and E is either a t space or a bornological space the condition implies that E is dense in the strong dual of a Frechet-Montel space. When $\mathscr{B}$ is the set of all compact sets and E is metrizable the condition implies that E is finite dimensional. In addition to proving these three propositions the author discusses various related questions. The word "fundamental" does not appear in the statement of proposition 1 but was presumably omitted by accident.

G. W. Mackey (Cambridge, Mass.).

Grothendieck, A. Erratum au mémoire: Produits tensoriels topologiques et espaces nucléaires. Ann. Inst. Fourier, Grenoble 6 (1955–1956), 117–120.

En produisant un contre-exemple, l'auteur montre que, dans sa Thèse [Mem. Amer. Math. Soc. no. 16 (1955); MR 17, 763], la partie 2° du lemme 13, ch. I, p. 130, est fausse; mais il signale que le rôle de ce lemme était de pure commodité: on peut l'éviter complètement dans la suite.

L'auteur profite de l'occasion pour signaler une erreur analogue dans un autre article [Canad. J. Math. 5 (1953), 129–173; MR 15, 438]. *J. Sebastião e Silva.*

Ehrenpreis, Leon. Cauchy's problem for linear differential equations with constant coefficients. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 642–646.

Dans cette étude l'auteur utilise des vues personnelles sur la transformation de Fourier de fonctions analytiques et de distributions [Amer. J. Math. 76 (1954), 883–903; MR 16, 834]. Soit $\mathscr{H}$ l'espace des fonctions entières $f(t,x)$ sur C^{n+1} et $\mathscr{H}_0$ celui des fonctions entières $g(x)$ sur C^n. On considère d'abord les opérateurs D réguliers, i.e. de la forme $D=\partial^m/\partial t^m+D_0(\partial^{m-1}/\partial t^m)+\cdots+D_m$, où les D_j sont des opérateurs différentiels en x à coefficients constants. Le problème de Cauchy consiste maintenant à trouver $f\in\mathscr{H}$ telle que

$$(\alpha)\qquad Df=g,\ \frac{\partial^k f(t,x)}{\partial t^k}\bigg|_{t=0}=g_k(x),\ k=0,\cdots,m-1,$$

avec $g\in\mathscr{H}$, $g_k\in\mathscr{H}_0$. L'auteur démontre le théorème suivant: le problème de Cauchy est bien posé (plus précisément, il admet une solution unique, qui dépend continûment de g et g_k), si, et seulement si, m est égal à l'ordre de D (on dit alors que D est parabolique).

En remplaçant $\mathscr{H}$ par l'espace $\mathscr{E}$ des fonctions indéfiniment dérivables, la même méthode permet de retrouver un résultat analogue de Gårding [Acta Math. 85 (1951), 1–62; MR 12, 831]. L'auteur envisage encore le cas d'opérateurs non-paraboliques et d'opérateurs irréguliers.

J. Sebastião e Silva (Lisbonne).

Ehrenpreis, Leon. Solutions of some problems of division. III. Division in the spaces, $\mathscr{D}'$, $\mathscr{H}$, $\mathscr{Q}_A$, $\mathscr{O}$. Amer. J. Math. 78 (1956), 685–715.

Soit A un espace de distributions (ou de fonctions) sur R^n (ou sur C^n). Le principal "problème de division" pour A se pose de la manière suivante: Etant donnés un opérateur différentiel D à coefficients constants et $T\in A$, est-il possible de trouver $S\in A$, telle que $DS=T$? Ou encore: La correspondence $S\to DS$ est-elle une application de A sur A? Malgrange et l'auteur avaient déjà trouvé, indépendamment, une réponse affirmative à cette question, dans le cas où A est l'espace des distributions d'ordre fini et dans d'autres cas encore [Ehrenpreis, même J. 76 (1954), 883–903; 77 (1955), 286–292; MR 16, 834, 1123]. Maintenant l'auteur généralise ce résultat au cas où A est l'espace $\mathscr{D}'$ de toutes les distributions et où D est un opérateur différentiel et aux différences finies, à coefficients constants. A cet effet, il emploie sa définition de la transformation de Fourier dans tout l'espace $\mathscr{D}'$ (ce qui revient à utiliser la transformation de Fourier dans $\mathscr{D}$) et il fait la caractérisation explicite de la topologie de l'espace **D**, image de Fourier de $\mathscr{D}$.

Pour l'espace $\mathscr{H}$ des fonctions entières on peut affirmer davantage: Pour toute $W\in\mathscr{H}'$ et toute $f\in\mathscr{H}$, on peut trouver $g\in\mathscr{H}$, telle que $W*g=f$. Ce problème aussi a été résolu au moyen de la transformation de Fourier définie dans le dual de l'espace $\mathscr{H}$ en question. Les mêmes méthodes réussissent dans plusieurs autres cas; l'auteur étudie en particulier les cas suivants: espace des fonctions entières d'ordre $\leq a$, espace des séries de puissances formelles, etc. *J. Sebastião e Silva* (Lisbonne).

Mikusinskiĭ, Ya. On the works of Polish mathematicians in the theory of generalized functions and in operational calculus. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 6(72), 169–172. (Russian)

Deprit, A. M. Une partition du spectre d'un endomorphisme continu d'un espace vectoriel topologique complexe. Nederl. Akad. Wetensch. Proc. Ser. A. 60= Indag. Math. 19 (1957), 55–59.

E is a linear topological space relative to the field of complex numbers, and u is a linear continuous transformation on E to E. Let $t(\lambda)=\lambda e-u$ where e is the identity. Then the points λ are divided into the distinct sets: (a) the resolvent set where $t(\lambda)$ is a bicontinuous automorphism of E, (b) the point spectrum $\sigma_p(u)$ where $t(\lambda)$ is not a monomorphism, (c) the continuous spectrum $\sigma_c(u)$ where $t(\lambda)$ is a continuous monomorphism but not an open map of E on $t(\lambda)(E)$, and (d) the residual set $\sigma_r(u)$ where $t(\lambda)$ is a continuous open monomorphism of E on $t(\lambda)(E)$ but is not an epimorphism on E. The last two definitions differ from those of Stone, where the continuous spectrum $\sum_c(u)$ consist, of the points where $t(\lambda)$ is a continuous non-open monomorphism of E on $t(\lambda)(E)$ such that $t(\lambda)(E)$ is dense in E, and the residual spectrum $\sum_r(u)$ consists of the λ such that $t(\lambda)$ is a continuous monomorphism of E on $t(\lambda)(E)$ such that $t(\lambda)(E)$ is not dense in E. If E is a linear normed (or metrisable, complete space then the following relations are announced:

$$\textstyle\sum_c(u)=\sigma_c(u)\cap\sigma_c({}^tu) \text{ and } \sum_r(u)=\sigma_r(u)\cup(\sigma_p({}^tu)\cap\sigma_v(u)),$$

where ${}^tu$ is the transpose of u. Moreover the Stone definitions also give a partition of the λ-plane. This latter is not always the case if E is a locally convex separated space.

T. H. Hildebrandt (Ann Arbor, Mich.).

Da Silveira, Miguel. General operational calculus in n variables. Portugal. Math. 15 (1956), 49–69.

L'auteur généralise, au cas de n variables, une partie d'un travail du reviewer sur le calcul opérationnel au point de vue des distributions [Portugal. Math. 14 (1956), 105–132; MR 18, 137]. Soit $\mathfrak{A}_k$ ($k=0, 1, \cdots$) l'espace des fonctions $f(z)$ de n variables complexes $z_1, \cdots, z_n$, holomorphes dans le produit Δ_k des demi-plans ouverts $\Re z_j>k$ et telles que la borne supérieure de $|f(z)/z_1^k\cdots z_n^k|$ dans Δ_k soit finie. En prenant cette borne pour norme $\|f\|_k$ de f, $\mathfrak{A}_k$ devient un espace de Banach et l'injection $\mathfrak{A}_k\to\mathfrak{A}_{k+1}$ est totalement continue. La limite inductive des $\mathfrak{A}_k$, que l'on désigne par $\mathfrak{A}_\omega$ (espace des fonctions holomorphes de croissance lente à droite), est donc un espace du type (LN^*) étudié par le reviewer [Rend. Mat. e Appl. (5) 14 (1955), 388–410; MR 16, 1122]. L'auteur détermine l'expression générale des applications linéaires continues de $\mathfrak{A}_\omega$ dans un espace localement convexe E, complet pour les suites. Après cela, il considère l'algèbre $\mathfrak{L}(E)$ des applications linéaires continues de E dans E, munie de la topologie de la convergence simple (supposée complète pour les suites) et il établit des conditions nécessaires et suffisantes (semblables à celles trouvées dans le cas où $n=1$), pour que, étant donné un système Θ de telles applications, $\Theta_1, \cdots, \Theta_n$, permutables deux à deux, il existe un homomorphisme continu $f\to f(\Theta)$ de l'algèbre $\mathfrak{A}_\omega$ dans $\mathfrak{L}(E)$, transformant la fonction z_j en Θ_j ($j=1, \cdots, n$) et chaque fonction constante c en cI. Dans ces conditions on a, univoquement, pour une convenable généralisation du concept d'intégrale et pour une valeur suffisante de k:

$$f(\Theta)=\frac{1}{(2\pi i)^n}\int_{k-\infty i}^{k+\infty i}\cdots\int_{k-\infty i}^{k+\infty i}\frac{f(\lambda_1, \cdots, \lambda_n)}{(\Theta_1-\lambda_1 I)\cdots(\Theta_n-\lambda_n I)}\,d\lambda_1\cdots d\lambda_n.$$

En particulier, l'algèbre $\mathfrak{A}_\omega$ est isomorphe (même topologiquement) à l'algèbre de convolution des distributions laplacisables, de support contenu dans le cône $t_i\geqq 0$, l'isomorphisme naturel étant l'inverse de la transformation multiple de Laplace. L'auteur se propose de développer dans un article ultérieur la théorie de cette transformation, d'après les méthodes directes du reviewer, tout en évitant les duals des espaces de distributions.

J. Sebastião e Silva (Lisbonne).

Leray, Jean. La théorie des points fixes et ses applications en analyse. Univ. e Politec. Torino. Rend. Sem. Mat. 15 (1955–56), 65–74.

Reprinted without change from Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. II, Amer. Math. Soc., Providence, R.I., 1952, pp. 202–208 [MR 13, 859]. *E. Begle* (New Haven, Conn.).

See also: Fourès-Bruhat, p. 743; Devinatz and Nussbaum, p. 748; S.-Nad', p. 748; Schreiber, p. 748; de Vito, p. 749; Mirkil, p. 757; Schröder, p. 765; Féron, p. 769.

Banach Spaces, Banach Algebras

★ **Riesz, Friedrich; und Sz.-Nagy, Béla. Vorlesungen über Funktionalanalysis.** Hochschulbücher für Mathematik, Bd. 27. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. xi+482 pp.

Translation by Siegfried Brehmer and Brigitte Mai, under the editorship of the former, from the French 2nd and 3rd editions of 1953 [MR 15, 132] and 1955 [MR 16, 837], respectively.

Kantorovič, L. V. Approximate solution of functional equations. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 6(72), 99–116. (Russian)

This survey is divided into three sections, "Gradient methods", "The method of Newton and other methods related to it", and "Convergence of processes of linear approximation". The bibliography lists 74 papers, 55 of these Russian. An introduction of nearly three pages emphasizes the importance of this application of functional analysis. As an annotated bibliography, the article will be of considerable interest to all those concerned with the field. *A. S. Householder* (Oak Ridge, Tenn.).

Jerison, Meyer. The set of all generalized limits of bounded sequences. Canad. J. Math. 9 (1957), 79–89.

A Banach limit is a linear form $\phi(x)$ on the space M of bounded sequences $x=\{\xi_n\}$ such that (1) $\phi(x)\geqq 0$ for $x\geqq 0$, (2) $\phi(Tx)=\phi(x)$, where $T\{\xi_n\}=\{\xi_{n+1}\}$, (3) $\phi\{1\}=1$. The existence of Banach limits may be derived from the Hahn-Banach theorem (this does not require topological considerations in M) or by studying topologies of the spaces M and M^*. Following the second way, the author proves the following. Let $T_n=n^{-1}\sum_{i=0}^{n-1}T^i$ and let $\psi\in M^*$ be a weak limit point of the forms of type ϕT_n with $\phi\in M^*$ satisfying (1), (3). Then ψ is a Banach limit. Let

Ω be the set of extreme points ω of the closed unit ball of M^* with property (1) for which $\omega(x)$ is independent of the value of any fixed ξ_n. Then the closed convex hull of the set of limit points of sequences ωT_n, $\omega \in \Omega$ coincides with the set of all Banach limits. Some known properties of almost convergent sequences [Lorentz, Acta Math. **80** (1948), 167–190; MR **10**, 367] are derived.
G. G. Lorentz (Ann Arbor, Mich.).

Vidav, Ivan. Quelques propriétés de la norme dans les algèbres de Banach. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 53–58.

Let B denote a complex Banach algebra with an identity and let H be the class of all $u \in B$ for which $\|1+i\xi u\|=1+o(\xi)$, for real $\xi\to 0$. If u and v are elements of H and ξ is real, then (1) $\|ae^{i\xi u}\|=\|a\|$ for every $a \in B$, (2) $\xi u \in H$, $u+v \in H$ and $i(uv-vu) \in H$, (3) H is closed, (4) $u+iv=0$ implies $u=v=0$, (5) $u\neq 0$ implies $\lim \|u^n\|^{1/n} > 0$, $n\to\infty$, (6) there exist real numbers α, β such that, for $\xi>0$, $\|e^{\xi u}\|=e^{\alpha\xi}$ and $\|e^{-\xi u}\|=e^{-\beta\xi}$. Using these properties, the author shows that, if for every $a \in B$ there exists $e^{i\alpha}$ (α real) such that $\|1+\xi e^{i\alpha}a\|$ admits a unique derivative at $\xi=0$, then either (a) B reduces to the scalars or (b) every $x \in B$ is of the form $x=\zeta_1 u+\zeta_2 v$, ζ_1, ζ_2 scalars, $u+v=1$ and $u^2=u$, the norm in B being given by $\|x\|=\max(|\zeta_1|, |\zeta_2|)$. A corollary is that B must reduce to the scalars if it is a Hilbert space. Also, if for every $a \in B$, $\|1+\xi a\|\,\|1-\xi a\|=1+o(\xi)$, $\xi\to 0$, then B reduces to the scalars. *C. E. Rickart* (New Haven, Conn.).

Yood, Bertram. Corrections to "Periodic mappings on a Banach algebra". Amer. J. Math. **78** (1956), 222–223.

These corrections were already included in the review of the original paper [same J. **77** (1955), 17–28; MR **16**, 719]. *C. E. Rickart* (New Haven, Conn.).

Singer, Ivan. Sur la représentation concrète des espaces de Banach. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 31–37. (Romanian. Russian and French summaries)

Soient E un espace de Banach, S la boule unité fermée du dual E' et A une partie de S. Pour tout $x \in E$ soit $\hat{x}$ la fonction définie sur A par l'égalité $\hat{x}(x')=x'(x)$. Dans cette note l'auteur donne: 1) quelques résultats concernant la représentation $x\to\hat{x}$; 2) quelques résultats sur les relations entre la dimension de E et certains ensembles de points extrémaux de S. Les résultats donnés sont élémentaires et en partie connus. *C. T. Ionescu Tulcea.*

Miyadera, Isao. On the representation theorem by the Laplace transformation of vector-valued functions. Tôhoku Math. J. (2) **8** (1956), 170–180.

The reviewer has given [Canad. J. Math. **6** (1954), 190–209; MR **15**, 620] necessary and sufficient conditions that a function $f(s)$ defined for $s>0$ into a reflexive Banach space X be represented as the Laplace transform of a function in $B_p((0,\infty), X)$ $(1<p\leq\infty)$. In the case $p=\infty$, the space X was restricted to be uniformly convex. These results were stated in terms of a particular integral inversion operator.

The author's principal result is to relax the condition of uniform convexity on X in the case $p=\infty$ to one of reflexivity. He also gives another proof of the representation theorem for the case $1<p<\infty$, in terms of the Widder-Post inversion operator, and shows by an example that the condition of reflexivity on X cannot be dropped. *P. G. Rooney* (Toronto, Ont.).

Citlanadze, È. S. The method of orthogonal trajectories for nonlinear operators of variational type in the space L_p. Amer. Math. Soc. Transl. (2) **5** (1957), 305–333.

A translation of the author's 1954 Russian paper reviewed in MR **16**, 934.

See also: Gel'fand, Raikov and Šilov, p. 714; Kadec, p. 733; Vertgeim, p. 734; Nöbeling und Bauer, p. 751; Kampé de Fériet, p. 769.

Hilbert Space

Livšic, M. S. On the spectral resolution of linear nonselfadjoint operators. Amer. Math. Soc. Transl. (2) **5** (1957), 67–114.

A translation of the 1954 Russian article reviewed in MR **16**, 48.

Devinatz, A.; and Nussbaum, A. E. On the permutability of normal operators. Ann. of Math. (2) **65** (1957), 144–152.

Let N_1 and N_2 be normal operators, not necessarily bounded, and let $\{K_1(z)\}$ and $\{K_2(z)\}$ be their corresponding resolutions of the identity. Then N_1 and N_2 are said to permute if $K_1(z_1)K_2(z_2)=K_2(z_2)K_1(z_1)$ for all complex z_1 and z_2. The principal result states that if N_1, N_2, and N are normal operators such that $N=N_1N_2=N_2N_1$, then N_1 and N_2 permute. Both the theorem and its proof are closely related to a previous result on the permutability of self-adjoint operators [A. Devinatz, A. E. Nussbaum, and J. von Neumann, Ann. of Math. (2) **62** (1955), 199–203; MR **17**, 178]. By employing this theorem the authors obtain an integral representation for weakly continuous semi-groups of normal (in general unbounded) operators over the parameter semi-group $I_m^+\times E_n^+$, where I_m^+ consists of the vectors with nonnegative integer components in an m-dimensional vector space and E_n^+ is the set of vectors in an n-dimensional euclidean space with nonnegative components. This generalizes a recent result by R. Getoor [Proc. Amer. Math. Soc. **7** (1956), 387–391; MR **18**, 221]. *R. S. Phillips* (Los Angeles, Calif.).

Sz.-Nad', B. Transformations of Hilbert space, positive definite functions on a semigroup. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6(72), 173–182. (Russian)

This paper is an exposition of a theorem of the author [Acta. Sci. Math. Szeged **15** (1954), 104–114; MR **15**, 719] on positive definite functions on a group to transformations of a Hilbert space, together with its applications to theorems, due to Halmos, the reviewer, Schaffer and Bram, on the representations of operators or groups of operators on a Hilbert space as contractions of operators on a larger space. With the exception of the discussion of Schaffer and Bram, the results are for the most part contained in the author's supplement to the third edition of "Leçons d'analyse fonctionelle" [Gauthier-Villars, Paris, 1955; MR **16**, 837]. *J. L. B. Cooper* (Cardiff).

Naĭmark, M. A.; and Fomin, S. V. Continuous direct sums of Hilbert spaces and some of their applications. Amer. Math. Soc. Transl. (2) **5** (1957), 35–65.

A translation of the 1955 Russian paper reviewed in MR **17**, 65.

Schreiber, M. Unitary dilations of operators. Duke Math. J. **23** (1956), 579–594.

Let A be a bounded linear operator on a Hilbert space

H such that $\|A\|\leq 1$. Such operators are called contractions. It is known [Halmos, Summa Brasil. Math. 2 (1950), 125–134; MR 13, 359; Sz.-Nagy, Acta Sci. Math. Szeged 15 (1953), 87–92; MR 15, 326] that there exists a Hilbert space K containing H as a subspace and operators P, U, where P is the projection of K onto H, and U a unitary operator on K such that for all $n\geq 0$ and $x\in H$, $A^n x=PU^n x$. The unitary operator U is called a unitary dilation of A, and K is a dilation space for A. Properties of unitary dilations and of dilation spaces are studied and, in particular, if A is a proper contraction (i.e. $\|A\|<1$) or a projection, minimal dilation spaces and the corresponding unitary dilations are characterized. If A is a proper contraction on a separable Hilbert space H, then K can be characterized as a space of functions $L_2(A)$ where the functions are defined on the unit circle C in the complex plane with values in H having inner product

$$[f,g]_A=\int_C (K(u)f(u),g(u))dm(u),$$

m the Lebesgue measure and $K(u)=\sum_{-\infty}^{\infty}\bar{u}^n A^n$, $A^{-|n|}=A^{*|n|}$. In this case the unitary dilation is identified as the operator $U(f)=uf(u)$. The spaces $L_2(A)$ are shown to be unitarily equivalent to $L_2(0)$, and $L_2(0)$ is shown to be a finite or countable direct sum of copies of the function space L_2 defined on C to the complex numbers. Thus for any A, canonical forms for K and U are given.

In case A is a projection, the unitary dilation of A is characterized as a direct sum of an identity operator defined on a suitable subspace of K and an operator of the type described above.

R. E. Fullerton.

Vala, Klaus. Sur la puissance extérieure d'un espace linéaire. Ann. Acad. Sci. Fenn. Ser. A. I. no. 233 (1956), 36 pp.

This paper contains proofs of results, previously announced by the author, on the exterior power of a Hilbert space and linear transformations in it [C. R. Acad. Sci. Paris 242 (1956), 2499–2500; MR 18, 138]. As a preliminary, some algebraic results on the exterior power of a not necessarily finite-dimensional linear space are established.

J. H. Williamson (Belfast).

de Vito, Luciano. Sugli autovalori e sulle autosoluzioni di una classe di trasformazioni hermitiane. Rend. Sem. Mat. Univ. Padova 25 (1956), 144–175.

The paper gives a rather complete theory for eigenvalue problems connected with a class of hermitian transformations in Hilbert space, which contains as particular cases those concerned with boundary value problems for partial and ordinary differential equations.

The class of hermitian transformations satisfying the following conditions is considered. a) Any transformation E admits an inverse E^{-1}. b) A completely continuous transformation T exists, whose restriction to the range of E coincides with E^{-1}. c) The eigenvectors of T belong to the domain of E.

A complete proof of the convergence of the Rayleigh-Ritz method is given, with respect to the approximating eigenvalues and eigenvectors, and the Courant maximum-minimum properties stated for the eigenvalues of E. Then the two-sided approximation of eigenvalues is studied and some new theorems established. Numerical examples are given at the end of the paper, concerning the boundary problems for elliptic membranes and plates.

G. Fichera (Rome).

Tseng, Ya. Yu. Virtual solutions and general inversions. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 6(72), 213–215. (Russian)

If A^{12} is a linear closed right operator from a Hilbert space $\mathfrak{M}^1$ into a Hilbert space $\mathfrak{M}^2$, a virtual solution of the (possibly inconsistent) equation $\chi^1 A^{12}=g^2$, g^2 given, is a vector χ_0^1 which minimizes $\|\chi^1A^{12}-g^2\|$. The virtual solution of smallest norm is called the extremal virtual solution, which is unique. Criteria for the existence of (extremal) virtual solutions are given. Furthermore, criteria for the weak (norm) convergence of extremal virtual solutions of sequences of equations $\chi_n{}^1A^{12}=g_n{}^2$ are given.

A virtual inverse for A^{12} is an operator R^{21} in case $A^{12}R^{21}=p^1$ and $R^{21}A^{12}=p^2$, where $p^{1,2}$ are the projections on the ranges of R^{21} and A^{12} respectively. It is assumed that the range of each operator is a subset of the domain of the other. The existence of a unique, closed R^{21} for each closed A^{12} is alleged. Furthermore the R^{21} so produced is such that $\chi_*{}^1=g^2R^{21}$ is an extremal virtual solution of $\chi^1A^{12}=g^2$ (if, of course, g^2 is in the domain of R^{21}).

If g^2 is in the domain of $(A^{12})^*$, then a virtual solution exists if and only if $(\chi'A^{12})(A^{12})^*=g^2(A^{12})^*$ has a solution, and the virtual solution is $\chi_*{}^1=g^2(A^{12}(A^{12})^*)^{-1}$. No proofs are offered.

B. R. Gelbaum.

Plans, Antonio. Fundamental properties of hyperquadrics in projective space with a countable infinity of dimensions. Rev. Mat. Hisp.-Amer. (4) 16 (1956), 202–228. (Spanish)

By the projective space E_∞ the author means the space whose elements are straight lines through the origin in the real Hilbert space of square-summable real sequences $x=(\xi_n)_0^\infty$. If A is a symmetric bounded infinite matrix (a_{ij}), a polarity in E_∞ is defined by $\rho\eta_i=\sum_{j=0}^{\infty}a_{ij}\xi_j$ ($i=0,1,\cdots$). The corresponding hyperquadric is the locus $\sum^{\infty}_{i,j=0}a_{ij}\xi_i\xi_j=0$. The paper is largely concerned with elementary observations, illustrated by examples, about how situations may resemble or differ from what can occur in the finite-dimensional case. In the discussion the author refers to the scheme of classification which he used in an earlier paper on bounded matrices [Rev. Acad. C. Madrid 46 (1952), 273–302; MR 14, 768]. In this classification the matrix A falls into one of 8 classes depending on the relations between its rows, likewise for the columns, and also depending on whether or not A is bicontinuous in a certain sense.

A. E. Taylor.

See also: Džrbašyan, p. 729; Voronovskaya, p. 730; Lévy, p. 770.

TOPOLOGY

General Topology

Ghéorghiev, Ghéorghi Iv. Images continues de la ligne droite dans le cercle. Univ. d'Etat Varna Fac. Tech. Méc. Annuaire 4 (1948–1949), 251–289. (Bulgarian. French summary)

L'article se rattache à deux articles antérieures de l'A. [même Annuaire 2 (1946–1947), 111–154, 155–201]. On considère des transformations continues $t \to x(t)$ de la droite euclidienne R^1 en un espace métrique E, en particulier en un cercle K. L'ensemble des points $x(t)$ $(t \in R^1)$ constitue une trajectoire; la représentation $x(t)$ $(x \in E)$ de celle-ci est régulière, si pour chaque t_0, chaque $\varepsilon>0$ et chaque $T>0$ il y a un $\delta=\delta(t_0, \varepsilon, T)>0$ tel que les relations $|\tau|\leqq T$, $\rho(x(t_0), x(\tau_0))\leqq\delta$ impliquent $\rho(x(t_0+\tau), x(\tau_0+\tau))\leqq\varepsilon$. Une partition P de E en trajectoires est dite régulière si pour chaque trajectoire $x \in P$ l'implication précédente subsiste pour un même δ. L'A. démontre quelques propriétés des décompositions régulières P de K, en particulier: P contient nécessairement un élément constitué d'un seul point (Th. 3). Pour qu'un $e \in P$ d'une P régulière de K se réduise à un point ou soit une courbe fermée simple, il faut et il suffit que l'ensemble e soit fermé.

Đ. Kurepa (Zagreb).

Haupt, Otto. Sur la notion de courbe continue dépourvue de paratingentes parallèles. C. R. Acad. Sci. Paris 244 (1957), 297–299.

Haupt, Otto. Sur les figures des courbes planes sans paratingentes parallèles. C. R. Acad. Sci. Paris 244 (1957), 440–442.

Let $p(t)$, $a\leqq t\leqq b$, be a curve in E_2 such that $p(t)$ is not constant on any t-interval. A paratangent of $p(t)$ at t_0 is the limit of a convergent sequence of lines through points $p(t_i')\neq p(t_i'')$ with $t_i'\to t_0$, $t_i''\to t_0$. The papers investigate curves without parallel paratangents. Considering two paratangents as always distinct if they are carried by the same line but belong to different t's proves to be too restrictive, because it leaves only monotonically transversed convex arcs. Therefore the following identifications are introduced: Two paratangents at $t'\neq t''$ carried by the same line are considered as identical if $p(t')=p(t'')$ and if there are subintervals $I'\ni t'$, $I''\ni t''$ of $[a, b]$ and a Jordan arc T such that $t\to p(t)$ maps both I' and I'' on T. In addition, if the image of a subinterval I of $[a, b]$ under $t\to p(t)$ is a straight segment S, then all paratangents containing S and for which $p(t) \in S$ are considered as identical. Otherwise paratangents carried by the same line are considered as distinct. The curves without parallel paratangents are characterized. They are either stars consisting of segments or they consist of a finite number of convex arcs and certain simple and double hooks.

H. Busemann (Los Angeles, Calif.).

Michael, Ernest. Continuous selections. III. Ann. of Math. (2) 65 (1957), 375–390.

This is the third in a series of papers [for motivation and terminology not defined below, see Ann. of Math. (2) 63 (1956), 361–382; 64 (1956) 562–580; MR 17, 990; 18, 325]. For any metric space Y, any subset A of Y, and any $\varepsilon>0$, $N_\varepsilon(A)$ denotes the set of points of Y at distance less than ε from A. For any space X, a carrier $\phi: X\to 2^Y$ is continuous provided that for each $\varepsilon>0$, every $x_0 \in X$ has a neighborhood U such that for every $x \in U$, both $\phi(x_0)\subset N_\varepsilon(\phi(x))$ and $\phi(x)\subset N_\varepsilon(\phi(x_0))$. (Note that continuity of ϕ depends on the metric of Y.) Let (*) denote the statement: X is paracompact, Y is a complete metric space with metric ρ, S a uniformly (with respect to ρ) equi-LC^n family of nonempty closed subsets of Y, and $\phi: X\to S$ is continuous with respect to ρ.

The hypotheses of the main selection theorems of the present paper contain (*) and a dimensional restriction, while their conclusions in each case state that every selection for $\phi|A$, where A is an appropriately situated closed subset of X, can be extended to a selection for ϕ. Typical example: if (*), and $\dim X\leqq n+1$, and A is a weak deformation retract of A, then every selection for $\phi|A$ can be extended to a selection for ϕ. (A subset A of X is a weak deformation retract of X provided there exists a continuous r: $X\times[0, 1]\to X$ such that for all $x \in X$, $r(x, 1)=x$, $r(x, 0) \in A$, and $r(x, 0)=x$ for all $x \in A$.)

As an application of the selection theorems, the paper contains some sufficient conditions for certain objects to be fibre spaces. In an appendix, the effect of the hypothesis that the carrier ϕ is continuous on some of the theorems (in particular, the characterization theorems) of the earlier papers is discussed.

{As was the case with "Continuous selections II", some of the definitions given in the present paper require correction; see the review cited above.}

M. Henriksen.

Cassina, Ugo. Elementi della teoria degli insiemi. II. Insiemi connessi irriducibili. Period. Mat. (4) 34 (1956), 85–108.

An exposition of order relations in sets irreducibly connected between two points. This continues an earlier paper [Period .Mat. (4) 33 (1955), 193–214; MR 17, 651].

M. E. Shanks (W. Lafayette, Ind.).

Mamuzić, Zlatko. Sur la caractérisation des espaces uniformisables. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. 9 (1956), 11 pp. (Serbo-Croatian summary)

E, M étant deux ensembles, soit f une application de E^2 en M; pour une famille F telle que chaque $X \in F$ contient les points $f(a, a)$ $(a \in E)$, considérons pour chaque $a \in E$, $X \in F$ l'ensemble $W_X(a)=\{b|b \in E, f(a, b) \in X\}$. Si $W_X(a)$ est considéré comme un voisinage de a (les a, X étant variables), on obtient un espace dit de la classe $E(M_F)$. Pour qu'un espace soit uniformisable, il faut et il suffit qu'il soit de la classe $E(M_F)$ et qu'il vérifie la condition (u) que voici: Si $a, b, c \in E$, $X \in F$, il existe un $Y \in F$ tel que $f(a, b) \in y \wedge f(a, c) \in y\to f(b, c) \in X$ (Th. 1). Les espaces topologiques complètement réguliers sont caractérisés comme les espaces de la classe $E(M_F)$ vérifiant (u), F étant telle que l'intersection des éléments de F se réduit à l'ensemble des points $f(a, a)$ $(a \in E)$ (Th. 2).

Đ. Kurepa (Zagreb).

Foulis, Linda Falcao. Subsets of an absolute retract. Proc. Amer. Math. Soc. 8 (1957), 365–366.

A topological space X is called a CAR* if every continuous function from a closed subset of a normal space E into X can be extended continuously over all of E. (For metric X, this is the same as an AR=absolute retract.) It is proved that if X is a compact Hausdorff CAR*, then the space $S(X)$ of non-empty, closed subsets of X, with the usual topology, is itself a CAR*. The novelty of this result

is that X need not be metric; for metric X, it is a special case of a theorem of Wojdisławski [Fund. Math. **32** (1939), 184–192], who assumed only that X is connected and locally connected (rather than a CAR*). {Reviewer's note: The historical introductory paragraph is unfortunately misleading, and should be disregarded. In particular, the above theorem of Wojdisławski is erroneously credited to Strother [Duke Math. J. **22** (1955), 551–556; MR **17**, 288].} *E. Michael* (Princeton, N.J.).

Nöbeling, Georg; und Bauer, Heinz. Ergänzung zu unserer Arbeit "Über die Erweiterungen topologischer Räume". Math. Ann. **132** (1957), 451.

A small correction to the paper referred to in the title [Math. Ann. **130** (1955), 20–45; MR **17**, 390].
E. Hewitt (Seattle, Wash.).

Landsberg, Max. Filter mit endlichem Index und Linearformen auf Produkten R^J. Math. Ann. **132** (1956), 256–262.

A filter has finite index if it is an intersection of a finite number of ultra filters. Its index (always finite) is the number of distinct ultrafilters which contain it. In the first part of the paper various characterizations and properties of filters of finite index are studied. For example, a filter of finite index is principal [in the sense defined by J. Schmidt, Math. Nachr. **7** (1952), 359–378; MR **14**, 255] if and only if every ultra filter which contains it is principal. Let J be any set and let R^J denote the vector space of all real valued functions on J. In the second part of the paper the author associates a filter $\phi(u)$ with each non-trivial linear functional u on R^J and discusses the connection between the properties of u and $\phi(u)$. For example, if $\phi(u)$ has finite index then u is continuous (in the weak topology defined by the linear functionals associated with the points of J) if and only if every ultra filter containing $\phi(u)$ is principal.
G. W. Mackey (Cambridge, Mass.).

Ehresmann, Charles; et Weishu, Shih. Sur les espaces feuilletés: théorème de stabilité. C. R. Acad. Sci. Paris **243** (1956), 344–346.

A leaved or laminated structure on a set E consists of a pair of topologies T, T', with $T \subset T'$, such that the topologies induced by T and by T' on any T'-open set are identical. The T'-components of E are the leaves; the space formed by the leaves, with the decomposition topology, is the transversal space. The holonomy group of the lamination is defined by, speaking intuitively, transporting the local transversal space (obtained from the induced lamination of T-open sets) around closed paths in a leaf. The stability theorem asserts that under certain hypotheses, in particular finite holonomy group, a compact leaf has a base of neighborhoods which are all unions of compact leaves. *H. Samelson.*

Burgess, C. E. Separation properties and n-indecomposable continua. Duke Math. J. **23** (1956), 595–599.

A continuum M is n-indecomposable (indecomposable under index n) if n is the largest integer such that M is the union of n continua no one of which is contained in the union of the others. The following two results, as well as several related results, were proved by the author.

(1) If M is a bounded plane continuum such that no pair of subcontinua of M separates M and no subcontinuum of M cuts M between two open subsets of M, then there is a positive integer n less than five such that M is n-indecomposable.

For a point x in a (compact metric) continuum M, denote by Nx the set of points p of M (including x) such that M is not aposyndetic at p with respect to x. Let K be the collection of such subcontinua, Nx, of M. F. B. Jones has shown that Nx is a continuum [Amer. J. Math. **70** (1948), 403–413; MR **9**, 606].

(2) In order that, for a compact metric continuum M, the collection K should be finite, it is necessary and sufficient that there exist a positive integer n such that M is n-indecomposable. *R. W. Bagley.*

de Vito, Luciano. Sugli autoomeomorfismi periodici di una striscia. Rend. Sem. Mat. Univ. Padova **26** (1956), 124–138.

La trasformazione t, della striscia S, compressa fra le orizzontali di equazioni $y=0$ ed $y=1$ del piano reale euclideo (x, y), su se stessa, sia priva di punti uniti, trasformi in sé ciascuna di quelle orizzontali e sia periodica nella x di periodo 1, vale a dire sia permutabile colla traslazione ordinaria che porta il punto (x, y) di S nel punto $(x+1, y)$. Inoltre tutte le linee contenute in S e periodiche rispetto alla x, con periodo uguale all'unità, incontrino le rispettive immagini nella t. Allora fra le curve semplici e aperte contenute in S ne esiston di quelle che congiungon quelle tali orizzontali e che non incontrano le rispettive immagini nella t e nelle potenze, diverse dall'identità, di quella traslazione ordinaria.

G. Scorza Dragoni (Padova).

Cronin, Jane. Some mappings with topological degree zero. Proc. Amer. Math. Soc. **7** (1956), 1139–1145.

L'A. considera una trasformazione piana dipendente da un parametro ε:

$$L_\varepsilon\colon (x_1, x_2) \to (x_1', x_2')$$

definita dalle equazioni

$$x_i' = P_{m_i}(x_1, x_2) + \tilde{P}_{m_i}(x_1, x_2) + S_i(x_1, x_2)\varepsilon^{n_i} + R_i(x_1, x_2, \varepsilon),$$

dove $\tilde{P}_{m_i}$ è un polinomio omogeneo di grado m_i, $\tilde{P}_{m_i}$ è una funzione continua con le sue derivate prime infinitesima di ordine superiore a m_i rispetto a $r=(x_1^2+x_2^2)^{\frac{1}{2}}$, S_i è una funzione continua non nulla nell'origine, R_i è infinitesima con ε di ordine superiore a n_i rispetto a ε. Scopo del lavoro è di assegnare delle condizioni sufficienti affinchè, detto S un cerchio con il centro nell'origine θ e di raggio abbastanza piccolo, esista un ε_0 tale che per $\varepsilon<\varepsilon_0$ sia: $\theta \in L_\varepsilon^{-1}(S)$ oppure: $\theta \notin L_\varepsilon^{-1}(S)$. Tali condizioni sono applicabili anche quando l'indice topologico di L_ε nel punto θ sia nullo e consistono in certe ipotesi relative alla posizione reciproca delle radici reali e di molteplicità dispari delle due equazioni $P_{m_1}(t, 1)=0$, $P_{m_2}(t, 1)=0$. *C. Miranda.*

Ore, Oystein. Graphs and subgraphs. Trans. Amer. Math. Soc. **84** (1957), 109–136.

In this paper the author's theory of subgraphs and deficiency functions [Duke Math. J. **22** (1955), 625–639; Ann. of Math. (2) **63** (1956), 383–406; **64** (1956), 142–153; MR **17**, 394; **17**, 1116; **18**, 143] is extended to finite loopless undirected graphs. In such a graph G let a non-negative integer $\kappa(v)$ be associated with each vertex v. The problem is to find a necessary and sufficient condition for the existence of a subgraph H having just $\kappa(v)$ edges incident with v, for each v. The author's solution is related to one given by the reviewer [Canad. J. Math. **4** (1952), 314–328, **6** (1954), 347–352; MR **14**, 67; **16**, 57] but is stated in terms of a single set of vertices instead of two.

If A is a set of vertices a vertex v is called unfilled, exactly filled or overfilled from A according as the number $\rho(A, v)$ of edges of G joining it to members of A is $<, =$ or, $>\kappa(v)$. The deficiency $\delta(A)$ of A is the sum of the numbers $\kappa(v)-\rho(A, v)$ over the vertices unfilled from A. The author considers the subgraph of G generated by the vertices of A overfilled from A. A component C of this is called odd if the sum over its vertices of the numbers $\kappa(v)$ is opposite in parity to the number of edges joining a vertex of C to a vertex of A not in C. The number of odd components is the inner restriction $r^{(i)}$ of A. There is also an outer restriction $r^{(0)}$ of A similarly defined in terms of the unfilled vertices of the complement of A. The author writes $\delta_e(A)=\delta(A)+r^{(i)}+r^{(0)}$. He shows that the required subgraph H exists if and only if $\delta_e(A)\geq 0$ for all A. He uses the method of alternating paths employed by most writers on the subgraph problem.

In a final chapter the author applies his theorem to the case in which $\kappa(v)$ is a constant fraction of the degree of v. This includes the classical case in which G and H are both regular. *W. T. Tutte* (Toronto, Ont.).

See also: Motzkin, p. 712; Reichelderfer, p. 723; Naîm, p. 729; Demidovič, p. 738; Szmydt, p. 741; Klee, p. 746; Postnikov, p. 753; Kruse, p. 754; Mazurkiewicz, p. 768.

Algebraic Topology

James, I. M. Multiplication on spheres. I. Proc. Amer. Math. Soc. 8 (1957), 192–196.

A homotopy-commutative multiplication on a space A is a continuous multiplication $(x, y)\rightarrow x\cdot y$ with two-sided identity, such that the two maps of $A\times A\rightarrow A$ given by $(x, y)\rightarrow x\cdot y$ and $(x, y)=y\cdot x$ are homotopic. The main result of the paper is that an n-sphere admits a homotopy commutative multiplication, if and only if $n=1$. This is a far-reaching generalization of the following previously known special results: 1. The unit quaternion-multiplication on S^3 is not homotopy commutative [H. Samelson, Comment. Math. Helv. 28 (1954), 278–287; MR 16, 389; and independently G. Whitehead, ibid. 28 (1954), 320–328; MR 16, 505]. 2. The Cayley multiplication on S^7 is not homotopy commutative [Sugawara, Math. J. Okayama Univ. 5 (1955), 5–11; MR 18, 59]. Finally, the author's theorem also generalizes the much more elementary result of this reviewer [Ann. of Math. (2) 57 (1953), 579–590; MR 15, 54] to the effect that a bonafide commutative multiplication exists on S^n if and only if $n=1$. The proof of the author's theorem rests on the following result: Suppose that S^n admits a multiplication, where $n>1$. Let $\gamma\in\pi_{2n+1}(S^{n+1})$ be an element of Hopf invariant unity, and let $w=[i, i]\in\pi_{2n+1}(S^{2n+1})$. Then an element $\alpha\in\pi_{2n}(S^n)$ exists, which has a nonzero generalized Hopf invariant, and such that $w+2\gamma=E(\alpha)$. [This generalizes a result of P. J. Hilton and J. H. C. Whitehead, ibid. 58 (1953), 429–442; MR 15, 642.] *R. Bott.*

Hurewicz, Witold; and Fadell, Edward. On the spectral sequence of a fiber space. II. Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 241–245.

In an earlier paper [same Proc. 41 (1955), 961–964; MR 17, 520] the authors showed that if (X, B, p) is a fiber space with fiber F and the base space B is $(r-1)$-connected for $r>1$, then in the associated spectral sequence for singular homology the terms $E_2, E_3, \cdots, E_r$ are all isomorphic. Therefore E_r can be identified with $H(B, H(F, G))$, where G is the coefficient group.

In the present paper it is shown that under the above identification the boundary operator d_r in E_r corresponds to cap product with the characteristic cohomology class of B, $\gamma\in H^r(B, \pi_r(B))$. The cap product in question is based on a pairing of $\pi_r(B)$ and $H_n(F, G)$ to $H_{n+r-1}(F, G)$. To define this pairing it is first shown how the fiber space structure gives rise to a continuous map $\Lambda\times F\rightarrow F_\Lambda$ (where Λ is the loop space of B), which defines a pairing of $H_{r-1}(\Lambda)$ and $H_n(F, G)$ to $H_{n+r-1}(F, G)$. Then the desired pairing results on applying the homomorphisms

$$\pi_r(B)\rightarrow\pi_{r-1}(\Lambda)\rightarrow H_{r-1}(\Lambda).$$

E. H. Spanier (Chicago, Ill.).

Brahana, Thomas R. A theorem about local Betti groups. Michigan Math. J. 4 (1957), 33–37.

Let S be a locally compact space, X and A closed subsets of S such that $X\supset A$, and x a point of A. Then $(x: X, A)$ is called a local pair, and groups $LH^n(x: X, A)$ are defined in a natural manner by the direct limit method, as well as corresponding homology sequences, the latter being proved exact. Moreover, if a local triad $(x: X; X_1, X_2)$ be defined consisting of x, X and closed sets X_1, X_2 containing x then Mayer-Vietoris sequences in a point may be set up and proved exact. An application is made to the theory of generalized manifolds (gm); specifically, it is shown that if M_1 and M_2 are n-gms with boundaries K_1 and K_2 respectively, and $M_1\cap M_2\subset K_1\cap K_2$ where $M_1\cap M_2$ is an $(n-1)$-gm, then $M_1\cup M_2$ is an n-gm with boundary. This generalizes an earlier result of P. A. White [Ann. Scuola Norm. Sup. Pisa (3) 4 (1950), 231–243; MR 13, 55]. *R. L. Wilder.*

O'Neill, Barrett. A fixed point theorem for multi-valued functions. Duke Math. J. 24 (1957), 61–62.

The following theorem is proved: Let X be a compact ANR, in n-dimensional Euclidean space, which is homologically trivial. Let F: $X\rightarrow X$ be a set-valued function having the following properties, 1) $H_q(F(x))=0$ $(0\leq q\leq n-2)$, and 2) for each $x\in X$ and each neighborhood U of $F(x)$, there is a neighborhood V of x such that if $y\in V$, then $F(y)\subset U$ and in addition each $(n-1)$-cycle on $F(x)$ is homologous in U to a cycle on $F(y)$. Then F has a fixed point.

This is a generalization of results of O. H. Hamilton [same J. 14, (1947), 689–693; MR 9, 197]. A counterexample is presented to one of Hamilton's theorems, which asserts the existence of a fixed point when condition 2) above is weakened to upper semicontinuity.

E. G. Begle (New Haven, Conn.).

Capel, C. E.; and Strother, W. L. A theorem of Hamilton: counterexample. Duke Math. J. 24 (1957), 57.
Hamilton, O. H. A theorem of Hamilton: correction. Duke Math. J. 24 (1957), 59.

Capel and Strother present an example to show that Theorem 1 of a paper by Hamilton [same J. 14 (1947), 689–693; MR 9, 197] is incorrect. Hamilton points out that by a slight strengthening of the hypotheses of this theorem, the remaining results in his paper are saved.

E. G. Begle (New Haven, Conn.).

Aoki, Kiyoshi; Honma, Eiitirô; and Kaneko, Tetuo. On natural systems of some spaces. Proc. Japan Acad. 32 (1956), 564–567.

This paper presents without proofs certain results on Postnikov natural systems [Dokl. Akad. Nauk SSSR

(N.S.) **76** (1951), 359–362, 789–791; Trudy Mat. Inst. Steklov. **46** (1955); MR **13**, 374, 375; **17**, 652].

The main concern is with the construction of two suitably related systems to serve as natural systems of a simply-connected space and its loop space. It appears to the reviewer that the guiding geometric notion is the following: if X is obtained from Y by killing π_i, $i \geqq m$, then there are Postnikov invariants

$$k_{m+1} \in H^{m+1}(X; \pi_m(Y)),\ k_m' \in H^m(\Omega X; \pi_{m-1}(\Omega Y)),$$

and, with suitable conventions, k_m' is the supsension of k_{m+1}.

Let (G_i, k_i), (H_i, l_i) be natural systems and let

$$\cdots \to F_i \to G_i \to H_i \to F_{i-1} \to \cdots \to F_1 \to G_1 \to H_1 \to 0$$

be an exact sequence. Algebraic conditions are given under which a fibre-space (E, X, F, p) exists such that the natural systems of E and X are isomorphic to (G_i, k_i) and (H_i, l_i) respectively and such that the homotopy sequence of the fibration coincides with the given sequence. *P. J. Hilton* (Manchester).

Brown, Edgar H., Jr. Finite computability of Postnikov complexes. Ann. of Math. (2) **65** (1957), 1–20.

The main results of this paper are the following: (1) If X is a simply-connected finite simplicial complex, then $\pi_n(X)$ is finitely computable for all n; (2) if X, Y are simply-connected finite simplicial complexes with finite homology groups, then there is a finite procedure for deciding whether they have the same homotopy type; (3) if X is a simply-connected finite simplicial complex with finite homology groups and Y is a finite simplicial complex, then there is a finite procedure for the homotopy classification of maps of Y into X.

The technique is to construct a Postnikov system for a complete semi-simplicial complex N. Let π be an abelian group and let $K(\pi, n)$ be the c.s.s. Eilenberg-Maclane complex; thus a q-simplex of $K(\pi, n)$ is a cocycle $u \in Z^n(\Delta_q, \pi)$, where Δ_q is the standard q-simplex. Let $E(\pi, n)$ be the c.s.s. complex whose q-simplexes are cochains in $C^n(\Delta_q, \pi)$ and let $\delta: E(\pi, n-1) \to K(\pi, n)$ be the coboundary. Any $A^n \in Z^n(N; \pi)$ determines a (simplicial) map $\hat{A}^n$: $N \to K(\pi, n)$, from which we construct a 'fibre-space' $P = P(N, \pi, A^n)$ over N with projection p. The author calls this a Postnikov construction. If N is finite in each dimension and π is finite, then P is finite in each dimension and P^q can be finitely constructed from N^q, π, and A^n.

This construction is iterated to form a Postnikov system. Thus we get, for a given c.s.s. complex N, a sequence of complexes $P_n(N)$, abelian groups $p_n(N)$, and simplicial maps g_n: $N \to P_n(N)$, where

$$P_n(N) = P(P_{n-1}(N), p_n(N), A^{n+1}),$$

A^{n+1} being, essentially, a Postnikov invariant. The sequence is, of course, defined inductively starting with $P_1(N)$=c.s.s. complex of a point, $p_1(N)=0$, and $g_1: N \to P_1(N)$ the evident map. This sequence has the property that, if N is simply connected, then $p_n(N) \cong \pi_n(|N|)$ and, if M is a minimal subcomplex of the singular complex of $|N|$, then $M^n \cong P_n(N)^n$ (here $|N|$ is the geometric realization of N). Now if also the homology groups of N are finite (in positive dimensions) then, by a theorem of Serre, the homotopy groups of N are finite; with a finiteness condition on N we infer that, for each q and n, $P_n(N)$ is finite in each dimension and $P_n(N)^q$, $p_n(N)$ and $g_n|N^q$ are 'finitely constructable'. Statements (2) and (3) above readily follow.

We cannot immediately infer statement (1) since there is no guarantee that $\pi_n(X)$ is finite. The author circumvents this by an ingenious modification of the Postnikov construction. Let λ be a subset of π and let $E(\pi, \lambda, n)$ be the subcomplex of $E(\pi, n)$ consisting of cochains taking values in λ. By restricting δ to $E(\pi, \lambda, n-1)$ we obtain a fibre-space $P(N, \pi, \lambda, A^n)$ over N with projection p (which is, in general, into N). Then $P(N, \pi, \lambda, A^n)$ is finite in each dimension if N is, provided only that λ is finite, and $P^q(N, \pi, \lambda, A^n)$ can be finitely constructed from N^q, π, λ, and A^n. The author shows that if π is finitely-generated, it is possible to choose λ appropriately as a finite subset of π in such a way that $P(N, \pi, A^n)$ and $P(N, \pi, \lambda, A^n)$ are homologically equivalent in low dimensions. By such choices an inductive procedure for defining, for each n and q, objects $P_{n,q}(N)$, $p_{n,q}(N)$ and $g_{n,q}$, analogous to those above, is given. In particular, $p_{n,q}(N)$ is finitely computable and $\pi_n(|N|) \cong p_{n,q}(N)$ if $n<q$. *P. J. Hilton* (Manchester).

Postnikov, M. M. Investigations in homotopy theory of continuous mappings. III. General theorems of extension and classification. Mat. Sb. N.S. **40(82)** (1956), 415–452. (Russian)

The present article completes the author's definitive version of three short papers [Dokl. Akad. Nauk SSSR (N.S.), **76** (1951), 359–362, 789–791; **79** (1951), 573–576; MR **13**, 374, 375]; the first two papers were expanded into book form and published as [Trudy Mat. Inst. Steklov. **46** (1955); MR **17**, 652]. The problem is to use the natural (Postnikov) system of an arcwise-connected space X to solve the extension and classification problems for maps of polyhedra P into X.

The first part of the present work is devoted to a general expository discussion of maps and homotopies in the category of semi-simplicial (c.s.s.) complexes. Let M be a minimal complex of $S(X)$, the singular complex of X, let N be a c.s.s. complex and let $|N|$ be a geometrical realization of N. Then a map of $|N|$ into X is said to be normal if the associated map $N \to S(X)$ maps N into M. It is shown that, in considering the extension and classification problems, attention may be confined to normal maps and normal homotopies (defined in an analogous way).

In the second part, the problems are solved. As in the 1951 paper, the basic notion in the solution is that of the cocycloid associated with a normal map. The natural system of X may be understood as a sequence of fibre projections

$$\cdots \to X_{p+1} \xrightarrow{f_p} X_p \xrightarrow{f_{p-1}} X_{p-1} \to \cdots,$$

where X_p is a c.s.s. complex obtained from M by 'killing off' π_i, $i>p$, and f_p is the projection induced by the Postnikov invariant $k_p \in Z^{p+2}(X_p; \pi_{p+1}(X))$. A map $\mu_p: P \to X_p$ such that $\mu_p^* k_p$ cobounds may be lifted into X_{p+1}. Moreover an equation

$$(E_p) \qquad \mu_p^* k_p + \delta c_{p+1} = 0,\ c_{p+1} \in C^{p+1}(P; \pi_{p+1}(X))$$

prescribes a lifting map $\mu_{p+1}: P \to X_{p+1}$. Then a cocycloid is a sequence $(\cdots, c_p, \cdots, c_1)$, $c_p \in C^p(P; \pi_p(X))$, such that c_1 is a cocycle and such that maps μ_p, $p=1, 2, \cdots$, exist satisfying (E_p). Then a cocycloid determines maps μ_p and hence a normal map μ and, conversely, a normal map μ determines maps μ_p and hence a cocycloid (there are canonical cochains $d_p \in C^p(X_p; \pi_p(X))$ with $f_{p-1}^* k_{p-1} = \delta d_p$: put $c_p = \mu_p^* d_p$). Then a normal map of P^p can be extended to P^{p+1} if and only if the c_p of the associated

cocycloid satisfies (E_{p-1}); and two normal maps of P are homotopic if and only if the associated cocycloids are cohomologous (in a natural sense).

The same problems are treated (using Postnikov systems but from the computational viewpoint) in the paper reviewed above. *P. J. Hilton* (Manchester).

★ **Kruse, Arthur H. Introduction to the theory of block assemblages and related topics in topology.** National Science Foundation Research Project on Geometry of Function Space. University of Kansas, Lawrence, Kansas, 1956. viii+306 pp.

"Many results on retraction (including deformation retraction) hold for both CW-complexes and ANRs (relative to the class of metrizable spaces). Thus there arises the problem of developing a unified theory of retraction which may be specialized to both CW-complexes and ANRs.... The setting for the theory lies in the concept of clean-cut block assemblage pair (defined in Chapter V)." (From author's preface.)

Let X be a topological space. A defining family $\mathscr{W}$ for X is a set of subsets of X such that $A\subset X$ is open if and only if $A\cap W$ is open in W for each $W\in\mathscr{W}$. A block assemblage is a pair $(X, \mathscr{B})$ where X is a space and $\mathscr{B}=\{B_\mu\}$ is a family of closed subsets of X indexed by the ordinals $\mu<$ some ordinal ν such that (a) $\mathscr{B}$ covers X and is a defining family for X and (b) if $S_\mu=\bigcup_{\lambda<\mu}B_\lambda$, $\mu<\nu$, then $B_\mu\cap S_\mu$ is closed in X and $\{B_\lambda\cap B_\mu\}_{\lambda<\mu}$ is a defining family for $B_\mu\cap S_\mu$. Thus a CW-complex is a block assemblage with the closed cells as blocks. A clean-cut pair is a couple (X, A) such that X is a metrizable space, A is a closed subset and strong deformation neighbourhood retract of X, and $X-A$ is an ANR (relative to the class of metrizable spaces). Thus (X, A) is clean-cut if X is an ANR and A a closed subset and ANR. Then $(X, A, \mathscr{B})$ is a clean-cut block assemblage pair if $(X, \mathscr{B})$ is a block assemblage, $A\subset X$, and $(B_\mu, B_\mu\cap(A\cap S_\mu))$ is clean-cut for each $\mu<\nu$.

As stated, this definition is achieved in Chapter V (p. 207) after a wealth of introducing material which certainly makes this report self-contained. In Chapter V a homotopy extension theorem for maps of $(X, A, \mathscr{B})$s is proved and applications to retraction properties are made. Thus it is shown that if (X, A) is 1-trivial and n-simple, $n=2, 3, \cdots$, then if A is not a (deformation) retract of X, $X\times\dot{I}\cup A\times I$ is not a (deformation) retract of $X\times I$.

Chapter VI is concerned with extending partial realizations (of polyhedra in X) and with lifting maps into complexes (if $f:X\to Y$, $g:X\to|K|$, $\beta:|K|\to Y$ with $f\simeq\beta g$, then g lifts f into $|K|$ modulo homotopy). Chapter VII is concerned with extensions of maps. In an appendix, results appearing in Chapters 4 and 5 are greatly strengthened by the use of the fact (pointed out to the author by Spanier) that, under very general hypotheses, if $A\subset X$ and $B\subset Y$, then $X\times B\cup A\times Y$ is a deformation retract of $X\times Y$ if it is a retract of $X\times Y$. Unfortunately the results of this appendix are vitiated by the false statement that, if C_n is free abelian for all n, then H_n is the direct sum of its torsion subgroup and a free abelian group. This in turn leads to an incorrect proof of the Künneth formula, and statements 4.5(i), 5.4, 5.5 and 5.6 require amendment (in 5.6 the conditions presumably are designed to ensure that $X\times B\cup A\times Y$ is a non-retract of $X\times Y$). The errors in 4.5(ii) and 4.6 are of a different kind, apparently due to unfamiliarity with the second version of the Künneth formula. *P. J. Hilton.*

Yokota, Ichiro. On the cells of symplectic groups. Proc. Japan Acad. 32 (1956), 399–400.

This paper is a correction of a previous one [same Proc. 31 (1955), 673–677; MR 17, 774] and gives a cellular decomposition of Sp(n) into 2^n cells. As a consequence the author states that Sp(n) has no torsion groups and its Poincaré polynomial is

$$(1+t^3)(1+t^7)\cdots(1+t^{4n-1}).$$

No proofs are given. *G. Papy* (Brussels).

Copeland, Arthur H., Jr. On H-spaces with two nontrivial homotopy groups. Proc. Amer. Math. Soc. 8 (1957), 184–191.

If Y is a space with base point y_0, then the problem of putting an H-space structure on Y is the problem of extending the folding map ϕ: $Y\vee Y\to Y$, given by $\phi(y, y_0)=\phi(y_0, y)=y$, to $Y\times Y$.

Suppose now that Y is a locally finite 1-connected CW-complex and that ϕ has been extended as far as the m-section of $Y\times Y$. The next obstruction to extending ϕ is $\gamma\in H^{m+1}(Y\times Y, Y\vee Y; \pi_m(Y))$. Let X be constructed from Y by adjoining cells (of dim $\geq m+1$) to kill the homotopy groups π_i, $i\geq m$, and let $k\in H^{m+1}(X; \pi_m(Y))$ be the Postnikov invariant. Then X has an H-space structure inducing $\psi^*:H^{m+1}(X)\to H^{m+1}(X\times X)$; let

$$\rho:H^{m+1}(X\times X)\to H^{m+1}(X\times X, X\vee X)$$

be the projection and let

$$i^*:H^{m+1}(X\times X, X\vee X)\to H^{m+1}(Y\times Y, Y\vee Y)$$

be induced by the embedding $(Y\times Y, Y\vee Y)\subseteq(X\times X, X\vee X)$. The author shows that

$$\text{(I)}\qquad i^*\rho\psi^*k=\gamma.$$

From (I) it follows that γ vanishes if and only if k is primitive. Suppose in particular that Y has only two non-vanishing homotopy groups, π_n and π_m, $1<n<m$. Then X is a $K(\pi_n, n)$ and k is the Eilenberg-Maclane invariant. Moreover, γ is the only obstruction so that it follows in this case that Y admits an H-space structure if and only if k is primitive. From an unpublished theorem of J. C. Moore it further follows that, if such a space Y admits an H-space structure then Y must be essentially a loop space.

The author also examines the relation between the existence of H-space structures and the vanishing of the Whitehead products and gives an example of a space, not admitting an H-space structure, in which Whitehead products vanish.

{The author explicitly restricts the discussion to locally-finite complexes; this appears to exclude complexes X defined above. Dowker has shown that products of countable CW-complexes are CW so that the category of countable complexes would be suitable for the discussion. Of course, for any CW-complex Y, one may retopologize $Y\times Y$ with the fine topology without affecting its homology or homotopy}. *P. J. Hilton.*

See also: Bourgin, p. 745; Singbal, p. 745; Michael, p. 750; Ehresmann and Weishu, p. 751; Hermann, p. 762; van de Ven, p. 762; Vesentini, p. 763; Serre, p. 765.

GEOMETRY

Geometries, Euclidean and other

Schütte, Kurt. Schliessungssätze für orthogonale Abbildungen euklidischer Ebenen. Math. Ann. **132** (1956), 106–120.

Als euklidische Ebene wird eine affine Ebene bezeichnet, in der eine Inzidenzrelation und eine symmetrische Orthogonalitätsrelation definiert ist, in der durch jeden Punkt zu jeder Geraden genau ein Lot existiert. In einer früheren Arbeit [Math. Ann. **129** (1955), 424–430; MR 18, 62] hat Verf. die Bedeutung des Satzes von den antiorthologen Vierecken untersucht; er lautet: wenn in zwei echten Vierecken P_1 bis P_4 und Q_1 bis Q_4 für 5 Permutationen $(jkmn)$ der Ziffern 1 bis 4 gilt, dass P_jP_k senkrecht steht auf Q_mQ_n, so gilt dies auch für die 6. Permutation (Satz 1). In der vorliegenden Arbeit untersucht Verf. einige Spezialfälle dieses Schliessungsatzes auf ihre geometrische und algebraische Bedeutung. Diese Schliessungssätze sind: der Höhenschnittpunktsatz im Dreieck (1a); der Fünfecksatz: schneiden sich vier Höhen eines Fünfecks in einem Punkt, so geht auch die 5. Höhe durch diesen Schnittpunkt (1b); der konditionale Höhenschnittpunksatz: wenn ein Dreieck ABC einen Höhenschnittpunkt hat, so hat auch jedes Dreieck mit den Ecken ABD einen Höhenschnittpunkt, wo D auf der zu AB gehörigen Höhe des 1. Dreiecks liegt (1c); der konditionale Fünfecksatz: schneiden sich vier Höhen eines Fünfecks A_1 bis A_5 in einem Punkt O und hat das Dreieck OA_1A_3 einen Höhenschnittpunkt, so geht auch die 5. Höhe des Fünfecks duch O (1d); der Trapezsatz: gelten in den Vierecken P_1 bis P_4 und Q_1 bis Q_4, in denen $P_1P_2\|P_3P_4$ und $Q_1Q_2\|Q_3Q_4$ ist, fünf der Relationen P_jP_k senkrecht auf Q_jQ_k, so gilt auch die 6.Relation (1e). Aus (1b) folgt (1a), aus (1a) folgt (1) und aus (1) folgt (1b). Die Sätze (1), (1a) bis (1d) stehen in Zusammenhang mit der Existenz gewisser orthogonaler Korrelationen in der durch die unendlichferne Gerade abgeschlossenen projektiven Ebene, speziell solchen, die involutorisch sind. Eine orthogonale Korrelation ist definiert als eine eineindeutige Abbildung der projektiven Ebene auf sich, in der einem Punkt eine Gerade, einer Geraden ein Punkt entspricht und jede eigentliche Gerade senkrecht steht auf der Verbindung ihres Bildpunktes mit dem Bildpunkt der unendlichfernen Geraden. — Der Satz (1e) dagegen steht in Zusammenhang mit der Existenz einer orthogonal-isomorphen Abbildung eines Dreigewebes (A, B) auf ein Dreigewebe (A', B'); dabei besteht ein Dreigewebe (A, B) aus den Geraden durch A, durch B und den zu AB parallelen Geraden, ferner aus allen Punkten, die nicht auf AB liegen. Eine orthogonalisomorphe Abbildung zwischen zwei Dreigeweben ist eine eineindeutige Abbildung von Punkt in Punkt, von Gerade in Gerade, die die Inzidenz erhält und jeder Geraden eine zu ihr senkrechte Gerade zuordnet. Das algebraische Äquivalent des Trapezsatzes ergibt einen Alternativkörper als Koordinatenbereich und einen Antiautomorphismus der additiven Gruppe mit gewissen schwachen multiplikativen Bedingungen. Gelten in einer euklidischen Ebene das Fano-Axiom, die Sätze (1), (1c), (1d), (1e), so ergibt sich eine Geometrie über einem Alternativkörper der Charakteristik $\neq 2$, in dem ein Antiautomorphismus $a \to \bar{a}$ existiert, dessen Fixelemente im Zentrum liegen, so dass $y = ax$ senkrecht steht auf $x = (k\bar{a})y$, wo $k \neq 0$ ein Fixelement ist. *R. Moufang* (Frankfurt).

★ **Hilbert, D.; i Cohn-Vossen, S. Geometria poglądowa. [Intuitive geometry.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1956. 319 pp. 24.50 zł.

Translation under the editorship of Stefan Kulczycki of the authors' 1932 edition of Anschauliche Geometrie.

Neville, E. H. Notes on conics. No. 21. The oblique pedals of the focus. Math. Gaz. **41** (1957), 58–59.

Wichers, J. The nine-point circle and Steiner's ellipse. Simon Stevin **31** (1957), 83–85. (Dutch)

1. The common tangent lines of the nine-point circle and the three exscribed circles intersect the sides of the triangle in P, Q, R; theorem: AP, BQ and CR are concurrent lines. 2. The tangent lines touch Steiner's ellipse. *O. Bottema* (Delft).

ApSimon, H. G. Archimedean screws. Math. Gaz. **41** (1957), 38–40.

Polyakov, A. N. On the construction of the images of a regular icosahedron and dodecahedron. Rostov. Gos. Ped. Inst. Uč. Zap. no. **3** (1955), 111–116. (Russian)

Zverca, Sp. Some observations on elementary geometry. Gaz. Mat. Fiz. Ser. A. **9** (1957), 25–27. (Romanian)

Sispanov, Sergio. Formulas for areas and volumes bounded by closed contours and surfaces. Acta Cuyana Ingen. **1** (1954), no. 7, 1–6. (Spanish. German summary)

The paper develops heuristically the usual formulas for the entities described in the title. *L. M. Blumenthal.*

Bodnărescu, H. New theorems in connection with the theory of projections of plane angles. Gaz. Mat. Fiz. Ser. A. **8** (1956), 345–355. (Romanian)

Let $\sphericalangle B_1A_1C_1 = \omega_1$, and let $\sphericalangle BAC = \omega$ be the orthogonal projection of ω_1 on the plane P. The author is concerned with the case in which the angles ω and ω_1 are equal. One of the subcases, which the author considers, is that the angle ω_1 is acute and none of its sides are parallel to the plane P. The author proves for this subcase among others the following theorem: Let the angle $BAC = \omega < \frac{1}{2}\pi$ and a side A_1B_1 of the angle ω_1 be given. There are two angles $B_1A_1C_1$ and $B_1A_1C_1'$ respectively, and one angle $B_1A_1C_1$ (equal to ω), according to whether the side A_1B_1 and the normal of the plane P form an angle larger or less than ω. *B. Germansky* (Jerusalem).

Tummers, J. H. Points remarquables, associés à un triangle. Nieuw Arch. Wisk. (3) **4** (1956), 132–139.

Etant donné le triangle ABC et deux points L_1 et L_2; $A_1B_1C_1$ soit le triangle formé par les pieds de AL_1, BL_1, CL_1; l'intersections de AL_2, BL_2, CL_2 avec B_1C_1, C_1A_1, A_1B_1 sont P, Q et R; théorème: A_1P, B_1Q et C_1R passent par un point $(L_1, L_2) = L_3$. On a $(L_1, L_3) = (L_2, L_1)$. Un cas particulier: si L_1 soit le centre de gravité et L_2 l'orthocentre, L_3 est le point de Lemoine. Discussion de quelques lieux géométriques. *O. Bottema* (Delft).

Court, Nathan Altshiller. Cercles cosphériques. Mathesis **65** (1956), 516–518.

Si trois cercles tracés resp. dans les trois faces d'un

trièdre sont deux à deux cosphériques, ils sont sur une même sphère. *O. Bottema* (Delft).

Thwaites, B. An iterative construction for the trisection of a given angle. Math. Gaz. 41 (1957), 48–50.

Most Euclidean constructions for the trisection of a given angle depend on the construction and drawing of a curve. Slow iterative procedures of a naive kind are easy to concoct, but one cannot call to mind any which converge so rapidly that only one, or at most two, iterations are required.

The construction given is an iteration, the second or at any rate third step of which seems able to give the trisection correct to the accuracy associated with any drawing. An empirical variant of it gives even more rapid convergence. *Author's summary.*

van de Vooren, W. L. Composition of two rotations about concurrent axes. Simon Stevin 31 (1957), 73–79. (Dutch)

The paper is about finite rotations in space and the use of vectors. There is no mention of quaternions.

O. Bottema (Delft).

Černyaev, M. P. Example of an algebraic congruence of order three and class one. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 129–132. (Russian)

Bilinski, Stanko. Einige Anwendungen der Polarkoordinaten in der hyperbolischen Geometrie. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. 11 (1956), 25–35. (Serbo-Croatian summary)

To illustrate the superior utility of polar over anything resembling cartesian coordinates in hyperbolic plane geometry the author obtains the expression

$$\kappa=\frac{2\dot{r}^2\,\mathrm{ch}\,r+\mathrm{sh}^2\,r\,\mathrm{ch}\,r-\ddot{r}\,\mathrm{sh}\,r}{(\dot{r}^2+\mathrm{sh}^2\,r)^{3/2}}$$

for the curvature of a curve $r=r(\varphi)$ at the point (r,φ). (Dots indicate derivatives with respect to φ.) He shows that this expression is invariant under a change of origin, that it gives the classical formula when the radius of the hyperbolic plane tends to infinity, and that it gives the constant values 0 for a line, 1 for a horocycle, $\mathrm{cth}\,\rho(>1)$ for an ordinary circle of radius ρ, and $\mathrm{th}\,d$ (<1) for an equidistant curve distant d from its axis. *P. Du Val.*

★**Лобачевский, Н.И. [Lobačevskiĭ, N. I.] Три сочинения по геометрии. Вступительная статья А. П. Нордена. [Three works on geometry. Introductory article by A. P. Norden.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 415 pp. 14.60 rubles.

This book, published in commemoration of the death of Lobačevskiĭ a hundred years ago in 1856 contains three of his papers: 1. Geometry (1823), on Euclidean geometry, written at the eve of his discovery of the non-Euclidean geometry; 2. Geometrical investigations on the theory of parallel lines (1840), his famous paper on non-Euclidean geometry, originally written in German; 3. Pangeometry (1855), his last work, orginally written in French. The book contains an introduction on "Lobačevskiĭ's geometrical ideas" by A. P. Norden. To these three papers an extensive commentary by V. F. Kagan has been added.

H. A. Lauwerier.

Rozenfel'd, B. A.; and Levinov, A. M. Application of non-Euclidean geometry to certain problems of projective geometry. Trudy Sem. Vektor. Tenzor. Anal. 10 (1956), 249–258. (Russian)

As is well known the first theorems of projective geometry were obtained by considering those properties of Euclidean geometry which remain invariant under central projection. Afterwards non-euclidean geometry was shown to be a special interpretation of projective geometry with respect to an absolute quadric. In the present paper the authors derive a series of theorems on projective geometry starting from the fact that a quadric of index l in n-dimensional projective geometry, e.g.

$$-x_0^2-x_1^2\cdots-x_l^2+x_{l+1}^2+\cdots+x_n^2=0$$

can be considered as the locus of points having the same distance to two mutually complete polar primes, when $x_0^2+x_1^2+\cdots+x_n^2=0$ is the absolute quadric.

E. M. Bruins (Amsterdam).

Arcidiacono, Giuseppe. Sui gruppi aggiunti dei gruppi delle rotazioni negli spazi a 3, 4, 5 dimensioni. Rend. Mat. e Appl. (5) 15 (1956), 140–152.

Dans un article précédent [Portugal Math. 14 (1955), 63–71; MR 17, 644] l'auteur a indiqué une méthode pratique pour le calcul de fonctions de matrices et il l'a appliquée à la détermination des transformations finies des groupes de rotations dans les espaces R^3, R^4 et R^5. Maintenant il fait le calcul des groupes adjoints de chacun de ces groupes. *J. Sebastião e Silva* (Lisbonne).

Arvesen, Ole Peder. Sur les projections axonométriques. Norske Vid. Selsk. Forh., Trondheim 29 (1956), 68–72 (1957).

In this paper axonometric projection is studied by means of analytic geometry. The author gives a simple proof for the following well-known result: There exists a plane π through the origin which is not parallel to the image plan and which has the property that every line segment in π has a congruent image. The author makes also a few remarks concerning orthogonal axonometry and the axonometric projection of a four-dimensional cube.

E. Lukacs (Washington, D.C.).

See also: Linnik, p. 719; Plans, p. 749; Petty, p. 760; Hölder, p. 775.

Convex Domains, Integral Geometry

Rešetnyak, Yu. G. On a generalization of convex surfaces. Mat. Sb. N.S. 40(82) (1956), 381–398. (Russian)

Let M be a set in E^n which is homeomorphic to a solid $(n-1)$-dimensional cube. M is called a δ-surface, $\delta>0$, if for every point $p\in M$ there is a system of spheres Σ_p of radius δ which pass through p and have the following properties: No point of M is an interior point of a sphere in Σ_p. The intersection of the interiors of the spheres in Σ_p contain interior points. If $p_i\in M$, $S_i\in\Sigma_{p_i}$, $p_i\to p$, $S_i\to S$, then $S\in\Sigma_p$. (A convex surface is a δ-surface for every $\delta>0$.) For a given point $p\in M$ there is a neighborhood U of p on M and a rectangular coordinate system $x_1,\cdots,x_n$ such that U can be represented in the form

$$(*)\qquad x_n=A(\delta)(x_1^2+\cdots+x_{n-1}^2)-f(x_1,\cdots,x_{n-1}),$$

where $f(x_1,\cdots,x_{n-1})$ is a convex function and $A(\delta)$

depends only on δ. Conversely, if a surface F can be represented in the form (*) and the angles of the normals of the surface $x_n=f(x_1, \cdots, x_{n-1})$ with the x_n-axis are bounded away from $\frac{1}{2}\pi$, then F is a δ-surface. The shortest connection on δ-surfaces are studied in detail. They possess one-sided tangents at all points which are interior points of M and these depend continuously on their point of contact in the same way as for convex curves.

H. Busemann (Los Angeles, Calif.).

Rogers, C. A. Two integral inequalities. J. London Math. Soc. 31 (1956), 235–238.

A closed sphere S in n-dimensional Euclidean space E_n is said to be obtained from a Lebesgue-measurable set E in E_n provided that S has its center at the origin and has measure equal to that of E.

Let $\rho_i(X)$ $(i=1, \cdots, k)$ be the characteristic functions of a finite number of Lebegue-measurable sets in E_n, and let $\rho_i^*(X)$ be the characteristic functions of the spheres obtained from the corresponding sets by spherical symmetrization. It is shown that

$$\iint \rho_1(X)\rho_2(Y)\rho_3(-X-Y)dXdY \leqq \iint \rho_1^*(X)\rho_2^*(Y)\rho_3^*(-X-Y)dXdY.$$

For convex sets, it is shown more generally that if c_{ij} $(i=1, \cdots, k; j=1, \cdots, m)$ are any real constants then

$$\int\cdots\int \prod_{i=1}^{k} \rho_i(\sum_{j=1}^{m} c_{ij}X_j)dX_1\cdots dX_m \leqq \int\cdots\int \prod_{i=1}^{k} \rho_i^*(\sum_{j=1}^{m} c_{ij}X_j)dX_1\cdots dX_m.$$

It is pointed out that in E_1 the first of the foregoing results is a special case of an inequality due to F. Riesz [same J. 5 (1930), 162–168] and the second is an immediate consequence of the result that the volume of a convex polyhedron in m-dimensional space is not diminished by the process of central symmetrization. It is then shown that the n-dimensional case of each result can be deduced from the 1-dimensional case by straightforward application of the method of Steiner symmetrization.

It is shown finally that both results can be extended to apply more generally to Lebesgue-measurable functions $\rho_i(X)$.

E. F. Beckenbach (Los Angeles, Calif.).

Knothe, Herbert. Contributions to the theory of convex bodies. Michigan Math. J. 4 (1957), 39–52.

Let K be a convex body in E_n and let $\rho_K(\mathbf{x})$ be a functional defined relative to K which is continuous in $\mathbf{x}$ and K, non negative, logarithmically convex, and m-homogeneous. Logarithmic convexity is defined by the inequality

$$\log \rho_{K_\theta}\{(1-\theta)\mathbf{x}_1+\mathbf{x}_2\} \geqq (1-\theta) \log \rho_{K_1}(\mathbf{x}_1)+\theta \log \rho_{K_2}(\mathbf{x}_2),$$

where $K_\theta=(1-\theta)K_1+\theta K_2$ and $0 \leqq \theta \leqq 1$. Homogeneity of mth order is defined by the identity

$$\rho_{\lambda K+a}(\lambda \mathbf{x}+a)=\lambda^m \rho_K(\mathbf{x}).$$

The author proves that the $(m+n)$th root of the integral $\int_K \rho_K dv$ is a convex function under convex combinations $K_\theta=(1-\theta)K_1+\theta K_2$. Using this result he obtains extensions and improvements of certain isoperimetric type inequalities due to Bonnesen and Bol. Several misprints appear but these are confusing rather than serious. For example the matrix 14) should not have the expression $(1-\theta)$ appearing in the off-diagonal elements, and x_K should read x_j or x_n in the Jacobian.

It is unfortunate that a standard notation seems not to be achieved. "Convex" as applied to functions in this paper is what many others but not all others have termed "concave". While neither term can be said to have intrinsic merit in relation to the geometry of the graphs of functions it is a nuisance to have this ambiguity.

P. C. Hammer (Madison, Wis.).

Knothe, Herbert. Inversion of two theorems of Archimedes. Michigan Math. J. 4 (1957), 53–56.

This note contains proofs of the following two theorems which are inverses of theorems of Archimedes. 1. If the curved surface of each right cylinder circumscribed about a convex body K is equal to the surface of K, then K is a sphere. 2. If the volume of each right cylinder circumscribed about a convex body K is 3/2 times that of K, then K is a sphere.

P. C. Hammer.

Mirkil, H. New characterizations of polyhedral cones. Canad. J. Math. 9 (1957), 1–4.

The author proves the following theorem. If a closed convex cone P has all its 2-dimensional projections closed, then P is polyhedral. Conversely, a closed convex polyhedral cone P has its projections (of all dimensions) closed. He applies this theorem to achieve a corollary on the positive extendibility of positive functionals defined on a linear subspace. The corollary reads: Let E be a polyhedrally ordered vector space, and let F be a subspace with the induced ordering. Then every positive functional f on F can be extended to a positive functional on E. Conversely, let E be a vector space ordered by a closed cone in such a way that the above positive extension property holds for all subspaces F and positive functionals f on F. Then the ordering of E is polyhedral.

P. C. Hammer (Madison, Wis.).

Stein, S. An application of topology to convex bodies. Math. Ann. 132 (1956), 148–149.

If A is a subset of E^n, a cap of A is that part of A lying on or to one side of an $(n-1)$-hyperplane. If A is furnished with a continuous strictly positive density, then $\Phi(A)$ denotes the center of gravity of A. Theorem: If K is n-dimensional and convex, if each support plane has only one point of contact, and if B^{n-1}, the boundary of K, is furnished with a strictly positive continuous density, then as C runs through all caps of B^{n-1}, $\Phi(C)$ fills K in a one-one manner.

The corresponding theorem for caps of K itself is also proved, and it is also shown that there is a non-degenerate cap C_0 of K such that $\Phi(C_0)=\Phi(C_0 \cap B^{n-1})$.

E. G. Begle (New Haven, Conn.).

★ **Сантало, Л. А. [Santaló, L. A.] Введение в интегральную геометрию [Introduction to integral geometry.]** Izdat. Inostr. Lit., Moscow, 1956. 184 pp. 8.10 rubles.

Translation, by M. G. Šestopal under the editorship of A. M. Lopšic and I. M. Yaglom, of the author's 1953 English edition [MR 15, 736]. A twelve-page supplement by I. M. Yaglom is entitled "Integral geometry on a manifold of lineal elements". The bibliography contains many titles added by the editors.

See also: Motzkin, p. 712; Kadec, p. 733; Bouligand, Choquet, Kaloujnine et Motchane, p. 758.

Differential Geometry

★ **Bouligand, G.; Choquet, G.; Kaloujnine, M.; et Motchane, L. Applications de la géométrie des distances à divers problèmes classiques de géométrie infinitésimale. Etudes pratiques d'accès à la recherche. A. Section des actualités géométriques.** Centre de documentation universitaire, 5, Place de la Sorbonne, Paris (V^e), 1956. 83 pp.

Contents: A general introduction by Bouligand to metric space methods in geometry; a comparison by Choquet of Hadamard's classical paper on surfaces with negative curvature with the reviewer's extension of these results to non-Riemann spaces and without differentiability hypotheses; a report by Kaloujnine on A. D. Alexandrov's solution of Weyl's problem for general convex surfaces; an outline by Motchane of the reviewer's approach to intrinsic area in Finsler spaces; finally and without a clear connection with the rest, an exposition by Felix (whose name is not mentioned in the list of authors) of the reasons for Lebesgue's interest in the theory of polyhedra. *H. Busemann* (Los Angeles, Calif.).

Volkov, Yu. A. On deformations of a convex polyhedral angle. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 209–210. (Russian)

Pogorelov, A. V. A new proof of rigidity of convex polyhedra. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 207–208. (Russian)

The first paper establishes: Let V, V' be two convex polyhedral angles of more than two faces, which, in the same cyclical order, are pairwise congruent. If e_i, e_i' ($i=1, \cdots, n>2$) are the unit vectors from the vertices of V, V' along corresponding edges and θ_i, θ_i' are the dihedral angles of V, V' respectively at e_i, e_i' ($\theta_i=\pi$ or $\theta_i'=\pi$ admitted), then the vector $\sum(\theta_i'-\theta_i)e_i$ is different from 0 and points into the spherical image of V unless V and V' are congruent.

The second paper deduces very simply from this result the following general form of Cauchy's theorem. Two intrinsically isometric convex polyhedra in E^3 are congruent as sets in E^3. *H. Busemann.*

Sen'kin, E. P. Unique determination of convex polyhedra. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 211–213. (Russian)

Zalgaller, V. A. On deformations of a polygon on a sphere. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 177–178. (Russian)

The same general form of Cauchy's theorem as in the preceding review is derived by Sen'kin from the following theorem proved by Zalgaller: Let two simple closed geodesic polygons lie on an open hemisphere and have, in the same cyclical order, sides of pairwise equal lengths. If the angles of one polygon do not exceed the corresponding angles of the second, then the two polygons are congruent. *H. Busemann* (Los Angeles, Calif.).

Bouligand, Georges. Analyse linéaire réelle et surfaces minima ou apparentées. C. R. Acad. Sci. Paris **240** (1955), 2103–2104.

Let the support function of a surface S be denoted by $h(u, v)$, where u is the colatitude, and v the longitude, of the normal to S. It is shown that S is a minimal surface, or surface for which the sum of the principal radii of curvature vanishes identically, if and only if the function $rh(u, v)$ is harmonic, where r, u, v are polar coordinates. Thus the study of minimal surfaces is reduced to the study of homogeneous harmonic functions of order 1.

The analyticity of minimal surfaces is an immediate consequence of the foregoing analysis.

Surfaces for which the sum of the principal radii of curvature is of constant sign are similarly related to superharmonic functions and subharmonic functions of this same type.

The Dirichlet problem for the support function of minimal surfaces is briefly discussed, and the study of related classes of surfaces S_α, for which the Laplacian of $r^\alpha h(u, v)$ vanished identically, where α is a constant other than 1, is suggested. *E. F. Beckenbach.*

Bouligand, Georges. Surfaces minima et apparentées. C. R. Acad. Sci. Paris **240** (1955), 2276–2278.

It is shown that, for geodesic distances $\rho<\pi/2$ on the spherical image of a minimal surface S, the function $h(u, v)$ defined in the preceding review satisfies the mean-value property

$$2\pi h_0 \cos\rho = \int_0^{2\pi} h(\rho, v)dv.$$

An extention to the class S_α of surfaces, also defined in the preceding review, is indicated, and some comments are made concerning the Dirichlet problem for the function $h(u, v)$. *E. F. Beckenbach* (Los Angeles, Calif.).

Bouligand, Georges. Sur la difficulté de construire par un procédé permanent les surfaces à courbures opposées. C. R. Acad. Sci. Paris **244** (1957), 419–422.

Convex surfaces generalize surfaces with positive curvature in a geometrically satisfactory way. The difficulties in finding an equally satisfactory concept for a general surface of negative curvature are discussed. Various possibilities are listed, for example: Give a family $y'=f(x, y)$ of curves in the (x, y)-plane and look for the surfaces $z=h(x, y)$ for which a family of asymptotic lines is projected into the given family. This leads to the partial differential equation $h_{yy}f^2+2h_{xy}f+h_{xx}=0$. A further analysis in another paper is promised.
H. Busemann (Los Angeles, Calif.).

Picone, Mauro. Il parametro monormale di una varietà regolare dello spazio euclideo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 705–711.

Leichtweiss, Kurt. Das Problem von Cauchy in der mehrdimensionalen Differentialgeometrie. III. Natürliche Gleichungen. Math. Ann. **132** (1956), 201–245.

Parts I, II of this paper appeared in Math. Ann. **130** (1956), 442–474; **132** (1956), 1–16 [MR 17, 1129; 18, 507]. Part III considers the problem of characterising m-dimensional submanifolds of n-dimensional Euclidean space by means of "natural equations", i.e. relations between geometric invariants of V_m and special parameters, for example, relations between curvature and torsion of a skew curve and arc length.

The main result is that every analytic submanifold of n-dimensional Euclidean space can be characterised by natural equations. In particular, every hypersurface with Riemannian metric can be characterised by means of a bilinear function involving elementary symmetric functions of the principal curvatures expressed in certain geodesic parallel coordinates. A similar characterisation is obtained for the relative differential geometry of a pair of hypersurfaces, and this includes as a special case results

of affine differential geometry associated with Blaschke and Salkowski. Moreover a complete and independent system of invariants is obtained in an algebraic fashion from the natural equations of a V_m in an n-dimensional Euclidean space. The vanishing of the non-trivial invariants of this system characterises the linear submanifolds.

The paper uses methods and results previously obtained in part I. *T. J. Willmore* (Liverpool).

Wunderlich, Walter. Contributi al problema delle lossodromiche doppie. Rend. Mat. e Appl. (5) **15** (1956), 24–35.

A loxodrome is a curve which intersects the planes through a fixed axis at a constant angle. The author mostly reviews his earlier results on double loxodromes [cf. Arch. Math. **6** (1955), 230–242; Österreich. Akad. Wiss. Math. Nat. Kl. S.-B. II **164** (1955), 17–34; MR **17**, 75, 1000]. In the last sections the double loxodromes with isotropic axes are determined. *P. Scherk.*

Schmeidler, Werner. Notwendige und hinreichende Bedingungen dafür, dass eine Raumkurve geschlossen ist. Arch. Math. **7** (1956), 384–385.

Das Problem ist von Fenchel formuliert worden. [Bull. Amer. Math. Soc. **57** (1951), 44–54; MR **12**, 634]. Verf. gibt zwei Matrixgleichungen an, die notwendig und hinreichend sind damit eine Raumkurve, die für jeden Parameterwert ein eindeutig bestimmtes begleitendes Dreibein besitzt und deren stetige Krümmung und Torsion der Periodizitätsforderung genügen, geschlossen ist.

O. Bottema (Delft).

Reade, Maxwell O. On certain conformal maps in space. Michigan Math. J. **4** (1957), 65–66.

A conformal sense-preserving homeomorphism of the solid 3-sphere with fixed center is a rotation. This is proved by elementary methods similar to the ones used for conformal and quasiconformal mappings of the plane. The point is that much weaker regularity assumptions are needed than in Liouville's classical theorem.

L. Ahlfors (Cambridge, Mass.).

Vincensini, Paul. Sur une transformation des surfaces minima. C. R. Acad. Sci. Paris **241** (1955), 153–154.

The author announces the following method of generating a double infinitude of minimal surfaces related to any given (not necessarily real) minimal surface (M). Let M be the vector from the origin to the variable point on (M), let N be the unit normal at this point, and let V be any fixed unit vector in space. Then the surface (P) generated by the terminal point of the vector P, with initial point at the origin, given by

$$P=M+\left\{\left(i\int N\times dM-M\right)\cdot V\right\}V,$$

where $i^2=-1$, is a minimal surface.

A geometric construction of (P) from (M) and V is described as a cylindrical projection of (M) in the direction V, and it is pointed out that the relation between (M) and (P) through V is a reciprocal one.

It is stated, further, that under the transformation the lines of curvature and the asymptotic lines of (M) correspond, respectively, to the asymptotic lines and the lines of curvature of (P), so that in particular if (P) is transformed in the same way relative to a second vector V' into a minimal surface (P') then the relation between (M) and (P') is a conformal one. *E. F. Beckenbach.*

Gallo, Elisa. Alcune proprietà dei sistemi (G) nello spazio. Boll. Un. Mat. Ital. (3) **11** (1956), 557–565.

Terracini has discussed systems (G) of plane curves, i.e. systems of ∞^3 curves defined by differential equations of the form $y'''=G(x, y, y')y''+H(x, y, y')y''^2$ [Univ. Nac. Tucumán. Rev. A. **2** (1941), 245–329; MR **4**, 54]. In the present paper the discussion is extended to systems (G) of curves in 3-space, i.e. systems of ∞^6 curves defined by systems of equations of the form

$$y'''=(Ay''+Bz''+C)y'',$$
$$z'''=(Ay''+Bz''+C)z'',$$

where A, B, C are functions of x, y, z, y', z'. Since the concepts and methods employed depend very much on the theory for the planar case, they cannot be described briefly. Two of the principal results can be stated as follows. (1) To each lineal element, consisting of a point A and a line a through A, there corresponds a certain satellite point P on a, and a certain satellite plane π through A. The configuration consisting of the superficial element (A, π) and the two lineal elements (A, a) and (P, a) is called a 3-element. It is shown that the theory of a system (G) of curves is equivalent to the theory of a system of ∞^5 3-elements. in which the 3-elements do not satisfy either of two quite special conditions. (2) Except in certain unusual cases, to each lineal element (A, a) there corresponds a quadric Q having the following property. Consider the curves of the family (G) which are tangent to a at A; and, for each of these curves, consider the conic which has fourth order contact with the curve at A. Then each of these conics is bitangent to Q.

L. A. MacColl (New York, N.Y.).

Marcus, F. Caractérisation géométrique des réseaux et surfaces E de Cartan. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi **1** (1954), 22–27. (Romanian. Russian and French summaries)

Terracini's 'linear projective element' of a congruence of straight lines Γ, introduced in 1933, is defined by the crossratio: $\theta=(F, \bar{F}, \bar{\pi}^*, \pi^*)=(\pi, \bar{\pi}, \bar{F}^*, F^*)$; where F, $\bar{F}$, π, $\bar{\pi}$ and F^*, $\bar{F}^*$, π^*, $\bar{\pi}^*$ are the focal points and planes of a line g and the neighbouring line g^* of Γ.

The following theorem is proved in this note: The necessary and sufficient condition for a surface to be of class E [E. Cartan, Bull. Sci. Math. (2) **68** (1944), 41–50; MR **7**, 78] is that it should admit a conjugate net such that: $\theta_1=\theta_{-1}$, where θ_1 and θ_{-1} are Terracini's linear projective elements of the congruences of tangent lines to the two families of the net. *R. Blum.*

Ruscior, Ştefania. Réseaux d'égalé torsion géodésique. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi **2** (1955), 87–94. (Romanian. Russian and French summaries)

Ruscior, Ştefania. Réseaux à torsions géodésiques égales et constantes. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), 1–7. (Romanian. Russian and French summaries)

In each point M of a surface Σ in the euclidean 3-space there exist two directions along which the geodesic torsion has a preassigned value $T(u, v)$. These directions determine, therefore, on Σ, a "net of equal geodesic torsion".

These notes contain a number of results concerning such nets, of which the following two may be mentioned: 1) If the net of parameter curves is of equal geodesic torsion then the mean curvature H is given by $H=-2D'/F$, where F and D' are the coefficients of the mixed terms in the first and second fundamental form of Σ. 2) If the net of parameter curves is of equal and constant geodesic torsion then Δ/F (where $\Delta=\sqrt{(EG-F^2)}$ satisfies a certain Laplace equation. *R. Blum.*

Marcus, F. Sur les suites F. Bul. Inst. Politehn. Iaşi (N.S.) 1 (1955), 11–23. (Romanian. Russian and French summaries)

This paper contains, in the main, two theorems which give conditions for a net R of a surface R to: 1) be such that the congruences of the tangents should be congruences F; and 2) generate a sequence F. These conditions involve relations between the invariants of the corresponding Laplace equation (and their derivatives). *R. Blum* (Saskatoon, Sask.).

Rembs, Eduard. Bemerkungen zur infinitesimalen Flächenverbiegung. Abh. Math. Sem. Univ. Hamburg 20 (1956), 178–185.

Let Δ denote infinitesimal isometric deformations of a convex surface S. 1) Let S have plane boundaries C and let δ_k denote the variation of their curvature. (i) The formula

$$\oint_C \delta_k \cdot q^{(v)} ds=0$$

is discussed. (ii) If $\delta_k\equiv 0$, S is rigid. (iii) If δ_k is prescribed, Δ is unique [(ii) and (iii) are equivalent]. (iv) If S is spherical and C is a circle, there exists a Δ with given δ_k. 2) Blaschke's integral formula is derived from Herglotz's [cf. Rembs, Math. Nachr. 7 (1952), 61–64, 387; MR 14, 203]. 3) One of the twelve Darboux surfaces associated with Δ is shown to be essentially identical with a surface studied by Liebmann [Math. Ann. 54 (1901), 505–517]. 4) The case is studied that this surface degenerates into a point. *P. Scherk* (Philadelphia, Pa.).

Mishra, R. S. Generalisations of Mainardi-Codazzi equations in a K_m-connected space. Tensor (N.S.) 6 (1956), 108–114.

L'auteur envisage un sous espace analytique complexe à n dimensions (C_n) plongé dans l'espace complexe (C_m) à m dimensions K_m-connecté.

Moyennant la considération d'un ensemble de $(m-n)$ congruences de courbes de C_m et de leurs conjuguées, il généralise, pour (C_n), les équations de Gauss et de Mainardi-Codazzi de la théorie des variétés, plongées dans un espace à un nombre quelconque de dimensions. P. Vincensini (Marseille).

★**Kahane, Arno. Elemente din teoria congruenţelor de drepte. [Elements of a theory of congruences of lines.]** Editura Tehnica, Bucureşti, 1956. 148 pp. 4.50 lei.

Das vorliegende Buch bringt die wesentlichen Dinge der differentiellen Liniengeometrie, wie sie auch in allen größeren Lehrbüchern der Differentialgeometrie behandelt werden. Es findet sich also darin eine Theorie der Regelflächen und der Geradenkongruenzen des R_3 mit den zugehörigen wesentlichen Begriffen wie Asymptotenlinien, Brennflächen, Laplaceschen Netze usw. Besonders ausführlich wird auf die Theorie der W-Kongruenzen und die damit zusammenhängenden Fragen der Flächentransformation und Verbiegung eingegangen. {Es fällt auf, daß bei der analytischen Behandlung auf den Vektorkalkül verzichtet wird und auch die Plückerschen Geradenkoordinaten erst auf der letzten Seite erwähnt werden.} *W. Burau* (Hamburg).

Petty, C. M. On the geometry of the Minkowski plane. Riv. Mat. Univ. Parma 6 (1955), 269–292.

The paper deals with the plane case of the geometry of Minkowski in the sense developed by H. Busemann. We will mention some of the subjects considered: a) Properties of the closed curves of constant Minkowski curvature κ; b) Frenet formulas for the Minkowski plane (which involve two curvature functions κ_1, κ_2 different from κ), with application to the involutes and evolutes of a given curve; c) calculus of Minkowski trigonometric functions; d) determination of curves with a given Minkowski curvature or given curvature functions κ_1, κ_2. In the last case, the condition of isometry of two curves S, S' with identical κ_1, κ_2 does not coincide with that of the existence of a motion of the Minkowski plane carrying S into coincidence with S'. *L. A. Santaló* (Buenos Aires).

Sancho de San Román, J. Twisted curves with constant affine width. Collect. Math. 8 (1955–1956), 85–98. (Spanish)

If $\Gamma(x=x(s))$ and Γ_1 $(x_1=x_1(s_1))$ are two curves in three dimensional space, and the parameters s, s_1 are the corresponding affine lengths, then the affine distance from x to x_1 is defined by the triple scalar product

$$V(s, s_1)=(x', x_1', x_1-x),$$

where $x'=dx/ds$, $x_1'=dx_1/ds_1$. The paper deals with the pairs of curves for which max V (for $s_1\in\Gamma_1$) does not depend on s (pairs of affine equidistant curves). Essentially the properties given are the following: a) Such pairs are characterized by the conditions $(x', x_1'', x_1-x)=0$ and $(x'', x_1', x_1-x)=0$; b) for every Γ there are infinite Γ_1 which form with Γ an affine equidistant pair. If Γ and Γ_1 coincide, the curve is said to be of constant width and the properties above apply. *L. A. Santaló.*

Švec, Alois. Déformations projectives des surfaces à réseau conjugué dans S_5. Czechoslovak Math. J. 6(81) (1956), 118–124. (Russian summary)

Dans Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 73 (1938), 443–459, M. A. Terracini a donné les conditions pour qu'une surface (F) de S_5 douée d'un seul réseau conjugué admette une déformation projective du 3ème ordre. Portant plus spécialement l'attention sur les surfaces (F) admettant en chaque point une pseudo-normale [c'est-à-dire pour lesquelles les variétés planes à trois dimensions osculatrices aux deux courbes du réseau conjugué de F en un point quelconque se coupent suivant une droite non située dans l'hyperplan osculateur à F au point envisagé], l'auteur présente les conditions de M. A. Terracini sous la forme géométrique suivante: Si les deux transformées de Laplace de (F) suivant son réseau conjugué sont de véritables surfaces, la condition nécessaire et suffisante pour que (F) soit projectivement déformable d'ordre trois est que les deux congruences rectilignes réalisant la transformation de Laplace de (F) dans les deux sens [congruences du réseau conjugué de F] soient des congruences W.

Si l'une des transformées de Laplace de F est une surface et l'autre une courbe, la condition nécessaire et suffisante de déformabilité d'ordre 3 de F est que la congruence

transformant F en une surface soit W, et qu'en un point quelconque M de la courbe C transformée de Laplace de (F) la variété plane à trois dimensions osculatrice à C soit dans l'hyperplan contenant le plan osculateur à C et le plan tangent à F au point correspondant. Si les deux transformées de Laplace de (F) sont des courbes, la condition nécessaire et suffisante de déformabilité projective d'ordre trois pour (F) est que, pour chacune des deux courbes transformées, la variété plane à trois dimensions osculatrice à la courbe soit dans l'hyperplan contenant le plan osculateur à la courbe envisagée et le plan tangent à (F) au point correspondant. *P. Vincensini.*

Švec, Alois. Problèmes d'existence de la déformation projective des surfaces de S_5 possédant un réseau conjugué. Czechoslovak Math. J. **6(81)** (1956), 125–138. (Russian summary)

Dans ce nouveau mémoire l'auteur, utilisant les méthodes du calcul différentiel exterieur de E. Cartan, revient sur les surfaces de S_5 douées d'un seul reseau conjugué et admettant une déformation projective d'ordre 3 étudiées dans le travail analysé ci-dessus, pour en préciser le degré de généralité. Il montre que, si les deux transformées de Laplace des surfaces envisagées ne sont pas dégénerées, ces surfaces dépendent de dix fonctions d'une variable, alors que le degré de généralité n'est que de sept fonctions d'une variable seulement si l'une des deux transformées de Laplace est une surface, l'autre se réduisant à une courbe. *P. Vincensini* (Marseille).

Decuyper, Marcel. Quadrilatère de Demoulin d'une surface. Rend. Sem. Mat. Messina **1** (1955), 120–142.

The quadric of Lie of a surface S at a point P of S ordinarily touches its envelope in four points besides the point P. The quadrilateral of Demoulin has these four points as vertices. The particular circumstances which give rise to degeneracies of the quadrilateral possess simple geometric interpretations which lead to numerous applications. The author gives a résumé of the investigations of these applications and of problems arising from the study of the quadrilateral attached to a general surface. He cites the works of Bompiani, E. Cartan, Decuyper, Wilczynski and presents a careful exposition of the results of Demoulin, Godeaux, Rozet, Calapso and Bol. *P. O. Bell* (Cupertino, Calif.).

See also: Haupt, p. 750.

Riemannian Geometry, Connections

Debever, R. Etude géométrique du tenseur de Riemann-Christoffel des espaces de Riemann à quatre dimensions. I. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 313–327.

In this paper is studied the decomposition of the Riemann-Christoffel tensor of a 4-dimensional Riemannian space. Both algebraic and geometric methods are used and results are interpreted geometrically in a 3-dimensional space C_3 with absolute quadric, and in the 5-dimensional space C_5 of lines in C_3. References are left to a second part which is to follow. *A. G. Walker.*

Debever, R. Etude géométrique du tenseur de Riemann-Christoffel des espaces de Riemann à quatre dimensions. II. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 608–621.

This is a continuation of part I [see the article reviewed above]. The decomposition of the Riemann-Christoffel tensor in a Riemannian 4-space is studied further, together with the geometrical representations in spaces of three and five dimensions. Curvature invariants of degree up to five are discussed in detail and tensor expressions for them are given together with a number of identities. *A. G. Walker* (Liverpool).

Mishra, R. S.; and Krishna, Shri. Generalisations of the congruences of curves in Riemannian space. Tensor (N.S.) **6** (1956), 125–131.

The authors discuss the properties of congruences which are such that through each point of a sub-space V_n immersed in a Riemannian space V_m there passes one curve of the congruence, introducing the concepts of lines of curvature and asymptotic lines in V_n with respect to such a congruence. The second part of the paper attempts to generalise the idea of the spherical representation of a congruence in a Euclidean 3-space to hyper-spherical representation of a congruence in a V_{n+1} and the authors concluded that this is possible only when the V_{n+1} is flat. *A. J. McConnell* (Dublin).

Blum, Richard. The Bianchi identities for the conformal curvature tensor. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 13–16.

In any Riemann space V_n, the conformal curvature tensor $C^h{}_{ijk}$ satisfies certain equations as a result of the Bianchi identities of V_n. The author shows that, if $n\geqq 4$, these equations may be written in the form $K^h{}_{ijkl}=0$ where $K^h{}_{ijkl}$ are linear in the covariant derivatives of the conformal curvature tensor. These identities in $C^h{}_{ijk}$, which bear a certain similarity to the classical Bianchi identities, are termed the "conformal Bianchi identities". A count of the number of independent identities is given. *A. Fialkow* (Brooklyn, N.Y.).

Singal, M. K.; and Behari, Ram. Characteristic lines of a hypersurface V_n imbedded in a Riemannian V_{n+1}. Proc. Indian Acad. Sci. Sect. A. **44** (1956), 53–62.

Les courbes caractéristiques d'une surface de courbure gaussienne positive sont celles du système conjugué de la surface pour lequel, en chaque point, l'angle formé par les deux courbes du système est l'angle minimum formé par les différents couples de directions conjuguées issues du point envisagé. Les auteurs étudient la généralisations de cette notion aux hypersurfaces V_n d'un espace V_{n+1} de Riemann. Ils montrent que les directions caractéristiques sont des combinaisons linéaires des couples de directions principales pour lesquelles les valeurs des courbures principales sont distinctes, et que pour les directions caractéristiques appartenant un faisceau déterminé par un tel couple de directions principales les courbures normales sont égales, leur valeur commune étant la moyenne harmonique des courbures principales correspondantes. Ils considèrent aussi les directions de V_n pour lesquelles le rapport de la torsion géodésique et de la courbure normale est un extrémum, et montrent que ces directions sont des directions caractéristiques. *P. Vincensini.*

Ledger, A. J. Symmetric harmonic spaces. J. London Math. Soc. **32** (1957), 53–56.

In a previous paper [same J. **29** (1954), 345–347; MR **15**, 986], the author defined a completely harmonic manifold by extending the local definition given initially by E. T. Copson and H. S. Ruse [Proc. Roy. Soc. Edin-

burgh 60 (1940), 117–113; MR 2, 20]. He then obtained a class of spaces which are symmetric and completely harmonic. The purpose of this paper is to show that all completely harmonic symmetric spaces with positive definite indecomposible Riemannian metrics belong to this class. Considerable use is made of E. Cartan's work on symmetric spaces. *E. T. Copson* (St. Andrews).

Chaki, M. C. On a type of tensor in a Riemannian space. Proc. Nat. Inst. Sci. India. Part. A. 22 (1956), 89–97.

In this paper it has been assumed that there exists in a Riemannian space V_n, with metric tensor g_{ij}, a tensor F_{ijkl} satisfying the identities $F_{ijkl}+F_{ijlk}=0$, $F_{ijkl}-F_{klij}=0$ and the properties of such a tensor have been studied in a V_2, V_3 and V_4. Defining a space V_n to be homogeneous with respect to a symmetric tensor a_{ij} if the metric tensor satisfies $a_{ij}=\sigma g_{ij}$, where σ is a function of the coordinates, it has been shown that every V_2 is homogeneous with respect to the tensor F_{ij}, where $F_{ij}=g^{mn}F_{imnj}$ and conditions both necessary and sufficient for a V_3 and V_4 to be homogeneous with respect to F_{ij} have been obtained. Further, building a tensor of the type F_{ijkl} out of an arbitrary affine connection $\Gamma_{ij}{}^{t}$, some theorems involving F_{ijkl} and $\Gamma_{ij}{}^{t}$ have been established for a V_2 and V_3. (The author's summary.) *V. Hlavatý.*

See also: Bouligand, Choquet, Kaloujnine et Motchane, p. 758; Auslander, p. 762.

Complex Manifolds

Auslander, Louis. Four dimensional compact locally hermitian manifolds. Trans. Amer. Math. Soc. 84 (1957), 379–391.

A n-dimensional real manifold is called locally hermitian if it has a complex manifold structure with a hermitian metric with curvature and torsion equal to zero. Its fundamental group can be naturally identified with a discrete subgroup π of the group of all regid motions of n-dimensional euclidean space. The intersection of π and the rotation group is called its holonomy group. Two such manifolds of the same dimension are called similar if their fundamental groups are conjugate in the group of all affine motions. The classification of 4 dimensional locally hermitian manifolds up to the similarity is given: All possible holonomy groups are cyclic groups of order 1, 2, 3, 4, and 6; each of the first and the last one provides one manifold and each of the rest provides two. It is also shown that if the 4 dimensional locally hermitian manifold is not a torus then its Poincaré polynomial is

$$1+2x+2x^2+2x^3+x^4.$$

M. Kuranishi (Chicago, Ill.).

Hermann, Robert. Compact homogeneous almost complex spaces of positive characteristic. Trans. Amer. Math. Soc. 83 (1956), 471–481.

This paper is concerned with the classification of almost complex manifolds M having non-vanishing Euler characteristic and admitting a transitive, compact Lie group of automorphisms. The author proves that such a manifold M is isomorphic with a direct product (in the sense of almost complex manifolds) of almost complex manifolds which admit transitive and compact simple Lie groups of automorphisms. The classification problem is then reduced to the determination of all the pairs (G, L), where G is a compact simple Lie group and L a closed subgroup of G of maximum rank such that G/L has an invariant almost complex structure. With the help of known results about subgroups of maximum rank, the author showed that either L is the centralizer of a toral subgroup of G, or (G, L) is one of the following twelve pairs: (G_2, A_2), $(F_4, (A_2)^2)$, $(E_6, (A_2)^3)$, $(E_7, A_2\times A_5)$, (E_8, A_8), $(E_8, A_2\times A_6)$, $(E_8, (A_4)^2)$, $(E_8, A_1\times A_2\times A_5)$, $(E_8, (A_2)^4)$, $(E_7, (A_2)^3\times T_1)$, $(E_8, A_1\times(A_2)^3\times T_1)$, $(E_8, (A_2)^3\times T_2)$. Here T_n denotes the n-dimensional toral group, and A_n, G_2, F_4, E_6, E_7, E_8 denote the compact simple Lie groups for their respective Cartan type with the lower index indicating the rank. In those twelve cases, the invariant almost complex structure has non-trivial torsion (i.e., not derived from complex structure).

The problem of reducibility of the linear isotropic group of G/L is also discussed when G is a compact simple Lie group, and L is the centralizer of a periodic element in G. *H. C. Wang* (New York, N.Y.).

Schwartz, Marie-Hélène. Classes de Chern des quadriques complexes. Bull. Sci. Math. (2) 80 (1956), 144–155.

Let Q^{2n} be a non-degenerate hyperquadric of complex dimension n. Let $c_r(n, p)$ be the value of the Chern characteristic class c_r on the $2p$-cycle defined by $Q^{2p}\subset Q^{2n}$, $n+1=p+r$. (The author used an earlier notation of the reviewer such that c_r is of topological dimension $2p=2(n-r+1)$; this notation is in slight disagreement with the one currently in use.) Then

$$c_r(n, p)=2\sum_k\binom{n-2k}{r-1}\ (k \text{ integral}, 0\leqq k\leqq \tfrac{1}{2}p).$$

It follows that, if A^{2p}, $p\leqq\frac{1}{2}n$, is a projective space of p complex dimensions in Q^{2n}, the value of c_r on A^{2p} is $\frac{1}{2}c_r(n, p)$. These results essentially give all the Chern characteristic classes of Q^{2n}. Proof is by explicit construction of certain vector fields. *S. Chern* (Chicago, Ill.).

Frölicher, Alfred; and Nijenhuis, Albert. A theorem on stability of complex structures. Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 239–241.

Let X be a compact differentiable manifold. Let J_t be a C^1 family of complex structures on X, and let $\theta_0(J_0)$ denote the sheaf of germs of vector fields on X holomorphic with respect to the complex structure J_0. The authors prove the following theorem: if $H^1(X, \theta_0(J_0))=0$ then, for all sufficiently small values of t, J_t is equivalent to J_0, i.e. there exists a differentiable homeomorphism ψ of X onto itself such that $J_t=\Psi^*J_0$.

An unpublished result of R. Bott states that

$$H^1(X, \theta_0(J_0))=0$$

if J_0 is a homogeneous Kähler structure on a manifold X with finite fundamental group. If follows that these structures are stable in the sense described above. In particular the usual complex structure on the complex projective space is stable. *M. F. Atiyah.*

van de Ven, A. J. H. M. Characteristic classes and monoidal transformations. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 571–578.

Let (A, B) be a pair of complex analytic varieties, of complex dimension a and b respectively, B being regularly embedded in A. The present paper deals with the pair of compact complex analytic varieties $(\tilde{A}, \tilde{B})$ obtained from

(A, B) by a "$\mathfrak{S}^{n,k}$-Prozess" of base B, and proves the formula

$$q_*(u^{a-b+i-1})=(-1)^{a-b+i-1}\bar{c}_i \ (i=0, 1, \cdots, b),$$

where $u \in H_{2a-4}(\tilde{B}, Z)$ is the dual to the Chern class of the complex normal bundle of $\tilde{B}$ in , $\tilde{A}q$ is the restriction to $\tilde{B}$ of the analytic mapping of A onto A, and $\bar{c}_0, \bar{c}_1, \cdots, \bar{c}_b$ are the inverses of the homology classes on B dual to the Chern classes of the complex normal bundle of B in A. Moreover, it is shown how the Chern classes of $\tilde{A}$ are to be found, when the Chern classes of A and B and the Chern classes of the complex normal bundle of B in A are known.

The proofs will be developed in full in a later publication and are based on results of Chern, of Borel and Hirzebruch and of Thom. The results obtained are linked with others previously given by B. Segre [Ann. Mat. Pura Appl. (4) **35** (1953), 1–127; **37** (1954), 139–155; MR **15**, 822, 1140; **16**, 511], concerning the case when A, B are algebraic varieties and the correspondence between A and $\tilde{A}$ is a dilatation of base B, transforming B in $\tilde{B}$. Here, by using an as yet unpublished result by A. Aeppli, it is also explained why, for an algebraic variety A, the "$\mathfrak{S}^{n,k}$-Prozess" and the dilatations yield analytically equivalent results; hence the name "monoidal transformation" (taken as an extension of "dilatation") is used throughout instead of "$\mathfrak{S}^{n,k}$-Prozess". *B. Segre* (Rome).

Vesentini, Edoardo. Campi di elementi lineari complessi sopra una varietà complessa compatta. Ann. Mat. Pura Appl. (4) 42 (1956), 325–379.

The author starts with a compact complex manifold M, of complex dimension m, and an assigned set of r $(1\leq r\leq m)$ meromorphic functions $F^1, \cdots, F^r$, defined globally on M, and seeks to determine the homology class of the cycle carried by the complex subvariety J of M, of dimension $r-1$, which is the locus of points of contact of level varieties of $F^1, \cdots, F^r$. It is assumed that these functions satisfy rather stringent conditions of regularity. (It may be pointed out that the existence of such functions implies severe restrictions on the nature of M).

The method of attack is to consider the bundle of ordered sets of linearly independent differential forms of type (1, 0), whose coefficients belong to the line bundles determined (in the sense of Kodaira) by the squares of $F^1, \cdots, F^r$. At all points of M other than points of J and points of indeterminacy of the functions $F^1, \cdots, F^r$ a cross-section of this bundle is defined. The essential difficulty in the problem lies in the fact that the points of indetermination, which from a subvariety of complex dimension $m-2$, preclude a direct use of obstruction methods. The author treats the problem by embedding each of these subvarieties in a small tubular neighbourhood in M, and then examining in detail the properties of the restriction of the bundle to these neighbourhoods. In the case $r=1$, which is formally simpler than the other cases, the number of points in the set J is the value, on the manifold $M-A'$ got from M by removing all internal points of the (small) tubular neighbourhood A of the locus L of points of indetermination of F^1, of the first characteristic class of the restriction to $M-A'$ of the bundle; and this in turn is equal to the value of this cocycle on M, which is otherwise known, diminished by the image in $H^{2m}(M, Z)$ under the natural embedding, of the relative cohomology class $c \in H^{2m}(A, N, Z)$, N the boundary of A, which is the primary obstruction in the interior of A to the extension of the section of the bundle defined on the boundary N. It is shown that C is the image, under the Gysin homomorphism, of the first characteristic class of the restriction of the bundle to L. Thus J can be expressed in terms of the divisor of F' and the Chern classes of M and L. When $r>1$ the work is formally more complex, but does not involve essentially new difficulties.

In the case in which M is an algebraic variety, and the meromorphic functions are rational, the results reduce to known formulae due to Eger [C. R. Acad. Sci. Paris **204** (1937), 217–219] and the reviewer [Proc. London Math. Soc. (2) **45** (1939), 410–424; MR **1**, 84], with canonical systems replacing Chern classes. *J. A. Todd.*

See also: Satake, p. 731.

Algebraic Geometry

Segre, Beniamino. Alcune applicazioni di una proprietà aritmetica delle quadriche. Rev. Un. Mat. Argentina 17 (1955), 231–250 (1956).

Let $\varphi=\sum_{i,j=1}^{2n} x_i x_j$ be a quadratic form of a projective space S_{2n-1} defined over a field Γ, of characteristic $p\neq 2$, and let D $(\neq 0)$ be its discriminant. The author has shown in an earlier work [Univ. e Politec. Torino. Rend. Sem. Mat. 9 (1950), 137–144; MR 12, 739] the following theorem: A necessary condition for $\varphi=0$ to contain some S_{n-1} defined over Γ is that $(-1)^n D$ be the square of an element of Γ. In this paper, various algebraic and geometric applications of this theorem are given. We can refer here to only a few of these.

a) Let f, g be two quadratic forms in four variables defined over Γ and such that the discriminant $D(u)$ of the quadratic form $f+ug$ is without multiple roots. Under this hypothesis, the elliptic curve $w^2=D(u)$ is a unirational transform (over Γ) of the quartic $f=0$, $g=0$ of which it is the jacobian curve.

b) As an application of the above result, the jacobian curve of the elliptic quartic $\mathscr{C}$: $w^2=\varphi(z)$ [$\varphi(z)$ a polynomial of degree four in z, with the discriminant not zero] is determined. The unirational transformation over Γ of the curve $\mathscr{C}$ into its jacobian exhibits the quartic and the sextic covariant of the polynomial $\varphi(z)$: in particular, the known relation between these covariants and $\varphi(z)$ is found, and geometrically interfered.

c) The considerations mentioned in b) are partially extended to the polynomials $\varphi(z)$ of degre $2\varphi+2$ $(\varphi>1)$ with coefficients in Γ and discriminant not zero. Now, the hyperelliptic curve $\mathscr{C}$: $w^2=\varphi(z)$ is considered together with a convenient projective model of its Wirtinger (or Kummer) variety U, and its jacobian variety J. From the consideration of the correspondence [1, 2] (over Γ) between U and J, the author is led to the study of the decompositions of $\varphi(z)$ as a sum of two squares:

$$\varphi(z)=[\sigma(z)]^2+[\tau(z)]^2$$

(σ, τ polynomials of degree $\leq\rho+1$). In a convenient extension of Γ, these decompositions are related (biunivocally mod a certain equivalence) to the self-residue (complete) $g_{\varphi+1}$ of the curve $\mathscr{C}$ such that their double series is not contained in the smallest linear series multiple of order $\varphi+1$ of the $g_2{}^1$ belonging to the curve $\mathscr{C}$.

F. Gherardelli (Florence).

Bilo, J. The $\{p; q\}$-curves on a quadratic surface. Simon Stevin 31 (1957), 86–89. (Dutch)

A $\{p; q\}$-curve is the complete intersection of the qua-

dric and a surface V of the degree p. Author determines the equation of V for a given curve. *O. Bottema.*

Godeaux, Lucien. Remarque sur la surface du quatrième ordre possédant un point double inflexionnel. Publ. Sci. Univ. Alger. Sér. A. 2 (1955), 241–245 (1957).

Godeaux, Lucien. Construction de surfaces algébriques dont le diviseur de Severi est quelconque. Rend. Sem. Mat. Univ. Padova 26 (1956), 10–17.

È noto che la superficie F^6 di Enriques (avente $p_a=p_g=P_3=0$, $P_2=1$) è immagine di un'involuzione priva di punti uniti appartenente ad una superficie Φ avente $p_a=P_4=1$; e L. Godeaux [Bull. Internat. Acad. Sci. Gracovie. Cl. Sci. Math. Nat. Sér. A. **1914**, 362–368], ha fatto notare che ciò spiega perché il divisore di Severi della F^6 sia $\sigma=2$. Infatti un sistema lineare $|C|$ di curve tracciate su Φ contiene due sistemi lineari parziale $|C_1|$ e $|C_2|$ appartenenti all'involuzione, ed a questi corrispondono sulla F^6 due sistemi lineari distinti $|\Gamma_1|$ e $|\Gamma_2|$ con $2\Gamma_1\equiv 2\Gamma_2$. Da questa osservazione il Godeaux è stato condotto alla costruzione di superficie algebriche con divisore di Severi qualunque, partendo da una superficie Φ sostegno d'una involuzione d'ordine p priva di punti uniti, e mostrando che un sistema lineare $|C|$ su Φ contiene k sistemi lineari parziali $|C_1|$, $|C_2|$, $\cdots$, $|C_k|$ $(1<k\leq p)$ appartenenti all'involuzione. A questi sistemi corrispondono sopra una superficie F immagine dell' involuzione dei sistemi lineari distinti $|\Gamma_1|$, $|\Gamma_2|$, $\cdots$, $|\Gamma_k|$ tali che $p\Gamma_1\equiv p\Gamma_2\equiv\cdots\equiv p\Gamma_k$. La superficie F ha divisore σ multiplo di p ed in generale (se il divisore di Φ è 1) esattamente p.

Nella nota presente l'Autore utilizza questi suoi risultati dimostrando che se nelle equazioni di una superficie di Steiner o d'una superficie di Veronese si sostituiscono alle coordinate correnti delle forme quadratiche linearmente indipendenti si ottengono superficie aventi rispettivamente i caratteri: $p_a=p_g=3$; $p^{(1)}=9$, $\sigma=2$; $p_a=p_g=55$, $p^{(1)}=289$, $\sigma=2$. *D. Gallarati* (Genova).

Babbage, D. W. A pencil of quadric primals and its associated system. J. London Math. Soc. 32 (1957), 1–6.

Given a pencil $\lambda S+\mu S'=0$ of quadrics in n dimensional projective space, the author shows how to determine a rational singly infinite system of quadrics, of index $n-2$, of the form $\sum \lambda^{n-2-r}\mu^r Q_r=0$, associated in an invariant way with the pencil. Each Q_r is of course a covariant quadric of S and S', but the essential point about the system is that it has invariance of a combinatorial character. When $n=3$ the author's system reduces to one discussed by the reviewer [Proc. Cambridge Philos. Soc. 43 (1947), 475–487; MR 9, 170]. *J. A. Todd.*

Galafassi, Vittorio Emanuele. La parte reale delle rigate astratte reali. Ann. Scuola Norm. Sup. Pisa (3) 9 (1955), 283–304 (1956).

F_p is a ruled surface of genus p whose generators are represented by a real curve C_p having some real circuits. It is shown that a real birational model of F_p can be constructed with an arbitrary number of real sheets, each sheet being of an arbitrary order of connexion, and some of them having arbitrarily chosen orientability or inorientability. The construction is to take a real nonsingular model of C_p in Euclidean space, with no real point at infinity, and construct the surface as locus of a circle whose centre is the current point of C_p and the square of whose radius is a real rational function λ on C_p (the plane of the circle is also rationally determined.) The real zeros and poles of λ divide each real circuit of C_p into an even number of segments in which λ is alternately positive and negative, except for circuits in which it is of the same sign throughout; and each positive segment, or positive whole circuit, gives rise to a sheet of the surface, whose connectivity and orientability depend on whether the sheet corresponds to a whole circuit, or to a segment bounded by two zeros, a zero and a pole, or two poles of λ. The connectivity of any sheet can then be arbitrarily increased by dilating a suitable number of its points. *P. Du Val* (London).

Marchionna, Ermanno. Sul gruppo fondamentale di una curva algebrica. Applicazioni alle superficie multiple prive di curva diramante. Ann. Mat. Pura Appl. (4) 41 (1956), 43–71.

The problem of the present paper is, given a surface F of order m in three dimensions, under what conditions can F be the seat of a μ-ple surface, without branch curve? F is first projected, sufficiently generally, into an m-ple plane with ordinary branch curve Φ, which (counted μ times) must also be the branch curve of the μm-ple plane, projection of the μ-ple F. The fundamental group of Φ in the plane is studied in great detail, and a diagrammatic representation of the group is obtained, by which it is claimed any given problem of the kind can be solved, though there seems to be little in the way of general concrete results. The chief one is that the general surface of order m is not the seat of any multiple surface without branch curve, but on the suitable impositions of $(m-1)^2$ or $m(m-2)$ nodes (according as m is odd or even) it becomes the seat of a double surface. (It is remarked that there are two distinct surfaces of order 4 with 8 nodes, one of which is and one is not the seat of a double surface.) It is suggested that by imposing other singularities we may obtain multiple surfaces with $\mu>2$, but the only example given is the familiar cubic surface with three binodes, seat of a triple surface without branch curve. The point does not seem to be made that these surfaces are not really unbranched, as there are isolated branch points at the singular points. *P. Du Val* (London).

Tibiletti, Cesarina Marchionna. Una rappresentazione topologica delle curve di una superficie. Rend. Sem. Mat. Univ. Padova 26 (1956), 18–35.

Soit une surface irréductible Φ représentée par un plan multiple avec courbe de diramation simple $\varphi(x, y)=0$. La donnée de la tresse algébrique de φ détermine le groupe des substitutions des diverses valeurs de $z(x, y)$ lié au groupe de Poincaré des cycles de la riemannienne du plan π $(z=0)$ dont on a retiré la courbe φ. Si l'on considère une courbe de Φ projetée en $f(x, y)=0$, cette courbe du plan multiple peut représenter plusieurs courbes distinctes de Φ. L'A. cherche à déterminer ces courbes. Ceci conduit à la généralisation de la notion de tresse, en considérant la tresse notée $T(f-P_\varphi)$ formée à partir de la tresse $T(f+\varphi)$ où l'on n'utilise que les traits relatifs aux points critiques de f et aux points P_φ, intersections de f et φ. Les cycles de la riemannienne se représentent par des ensembles connexes fermés d'arcs de tresse, c'est-à-dire un ensemble de parties d'un fil décrit par une détermination $y_i(x)$ lorsque x décrit un lacet de π_x, tel qu'en ses extrémités la réunion de ces arcs soit connexe (cycle de la tresse). Le résultat fondamental est que tout cycle de la riemannienne se réduit à un cycle de la tresse $T(f-P_\varphi)$. A partir de ce ré-

sultat l'A. donne un procédé pour déterminer les différentes courbes projetées sur f, et pour particulariser celle passant en un point donné de Φ. Sur chacune de ces courbes on peut déterminer les diramations de la $z(x, y)$, et par la formule de Zeuthen son genre. Deux exemples sont traités: cubique nodale tritangente à la conique diramante d'une quadrique image de deux cubiques gauches, si cubique non tritangente ou cubique elliptique projection d'une sextique irréductible; droite bitangente à la sextique diramante d'une surface cubique, image d'une droite et d'une conique, si droite tangente, cubique rationnelle, si droite non tangente, cubique elliptique.

B. d'Orgeval (Dijon).

Chisini, Oscar. Sul comportamento effettivo delle polari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 547–551.

The author proposes an elementary proof of the theorem of Vesentini [Ann. Mat. Pura Appl. (4) **34** (1953), 219–245; MR **15**, 58] that, if C is a plane curve with a singularity at a point O, then the general curve having the same singularity has the property that the multiplicities of successive points of the sequence at O on the general first polar are given by Enriques' law of alternation. There are certain points in the argument which are not clear to the reviewer.

J. A. Todd (Cambridge, England).

Serre, Jean-Pierre. Sur la cohomologie des variétés algébriques. J. Math. Pures Appl. (9) **36** (1957), 1–16.

Using the same methods, the author completes a number of points that were left open in his recent paper on coherent algebraic sheaves [Ann. of Math. (2) **61** (1955), 197–228; MR **16**, 953]. Letting X denote an algebraic variety and $\mathcal{F}$ a coherent algebraic sheaf on X the main results are: If $H^1(X, \mathcal{F})=0$ whenever $\mathcal{F}$ is a sheaf of ideals in the sheaf of local rings of X, then X is an affine variety. If X is arbitrary and $q>\dim X$, then $H^q(X, \mathcal{F})=0$. If X is complete then $H^0(X, \mathcal{F})$ is a finite dimensional vector space over the field of constants.

M. Rosenlicht (Evanston, Ill.).

d'Orgeval, B. A propos des surfaces algébriques se touchant le long d'une courbe et du nombre maximum de leurs points doubles. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 247–250 (1957).

A partir de la propriété des surfaces se touchant le long d'une ligne, d'y avoir un certain nombre de points doubles, on donne une formule asymptotique pour le nombre maximum de points doubles d'une surface algébrique d'ordre donné.

Resumé de l'auteur.

Cantoni, Lionello. Nuovi tipi di trasformazioni birazionali nella teoria delle varietà abeliane reali. Ann. Scuola Norm. Sup. Pisa (3) **9** (1955), 207–233 (1956).

L'A. definisce e studia alcuni tipi di trasformazioni delle varietà abeliane reali [le quali, a meno di trasformazioni birazionali, coincidono con la propria complessa coniugata; cf. S. Cherubino, Giorn. Mat. Battaglini (3) **12**(**60**) (1922), 65–94]. Dapprima le trasformazioni dette quasi immaginarie pure che mutano una simmetria S della varietà $v'\equiv\bar{v}+c$, $c+\bar{c}\equiv0 \pmod{\omega}$ in una simmetria della schiera associata $v'\equiv-\bar{v}+d$, $d-\bar{d}\equiv0 \pmod{\omega}$ e poi un loro caso particolare (trasformazioni semipseudoordinarie). Si tratta di estensioni dei concetti di trasformazioni quasi reali e, rispettivamente, pseudoordinarie (validi nel caso che la S sia mutata in un'altra S della stessa schiera) introdotti e studiati dal Cherubino. Nella seconda parte del lavoro, con riferimento alle superficie iperellittiche reali (varietà abeliane reali a due dimensioni) l'A. determina la classificazione di un tipo particolare di trasformazioni semipseudoordinarie (esistenti solo per particolari superficie iperellittiche). Vengono inoltre studiate le condizioni di esistenza delle trasformazioni semiordinarie e vien data la classificazione delle trasformazioni semipseudoordinarie di una superficie iperellittica che abbia anche trasformazioni semiordinarie.

M. Rosati (Roma).

See also: Deuring, p. 719; Lustig, p. 719; van de Ven, p. 762; Vesentini, p. 763.

NUMERICAL ANALYSIS

Numerical Methods

Schröder, Johann. Das Iterationsverfahren bei allgemeinerem Abstandsbegriff. Math. Z. **66** (1956), 111–116.

Sufficient conditions for convergence of the iteration process $u_{n+1}=Tu_n$ are formulated. The space R of admissible functions $(u, v, \cdots)$ is supposed to be a "complete metric space" with respect to a partially ordered linear space N of "norms" $\{\rho(u, v)\}$. The criterion for convergence is expressed in terms of a continuous linear positive operator P, called a "bound", which is defined over the space N of norms. The operator T (not necessarily linear) is said to have the bound P if $\rho(Tu, Tv)\leq P\rho(u, v)$ for all u, v in the domain D of the operator T. The main theorem states: If the operator T has a bound P such that $\sum_{i=1}^\infty P^{i-1}\rho$ converges for all ρ in N and if D contains all $w \in R$ such that $\rho(w, u_1)\leq\sigma$, where $\sigma=\sum_{i=1}^\infty P^{i-1}(P\sigma_0)$ and $\sigma_0=\rho(u_1, u_0)$ (for some element u_0 in D) then the sequence $\{u_n\}$ defined by $u_{n+1}=Tu_n$ $(n=0, 1, 2, \cdots)$ converges to an element u which is the unique solution of $u=Tu$ in D and which satisfies the error estimate $\rho(u, u_1)\leq\sigma$. The author indicates how the theoretical estimates may be realized in practical problems. Detailed applications of the theory are to be found in an earlier paper [Z. Angew. Math. Mech. **36** (1956), 168–181; MR **18**, 152].

E. Isaacson (New York, N.Y.).

Downing, A. C., Jr.; and Householder, A. S. Some inverse characteristic value problems. J. Assoc. Comput. Mach. **3** (1956), 203–207.

These are the problems: Given a finite Hermitian matrix A, find a real non-singular, diagonal matrix D such that: (i) $D^{-1}AD^{-1}$ has prescribed characteristic values, or (ii) $A+D$ has prescribed characteristic values.

Criteria for existence of solutions are not known. When a solution exists for (i), and when the prescribed characteristic values are distinct and non-vanishing, the authors present two iterative processes to solve (i). It is shown how to solve (ii) similarly. There is a generalization of (i) to several matrices A_μ and a non-diagonal D.

One of the iterations for (i) converges quadratically, and requires computing eigenvalues and eigenvectors of a matrix at each iterative step. The other iteration requires only eigenvalues, but converges only linearly.

Note: The lemma on p. 204 is stated on p. 75 of MacDuffee, "The theory of matrices" [Springer, Berlin, 1933].

G. E. Forsythe (Stanford, Calif.).

Clere, D. Sur le calcul par itération des modes propres d'ordre supérieur. Rech. Aéro. no. **54** (1956), 39–48.

The author expounds two variants of the classical Duncan-Collar iteration for computing eigenvalues of a finite matrix L [Phil. Mag. (7) **17** (1934), 865–909]. In the first variant one computes an eigenvalue λ_1 and corresponding eigencolumn $X_{(1)}$ by ordinary iteration, and then deflates L to a new matrix $L^{[1]}$ whose eigenvalues $\lambda_2, \cdots, \lambda_n$ are the same as those of L. The other eigenvalue of $L^{[1]}$ is 0. The eigencolumns of L can later be recovered from those of $L^{[1]}$.

In the second variant the deflated matrix $L^{[1]}$ is replaced by an augmented matrix $L^{(1)}$ of n rows and $n+1$ columns. Details are given for the reduction to $L^{(2)}$, $(L^{3)}$ etc., and for recovering the eigencolumns of L. While the second variant involves less arithmetic it appears to need more storage. The advantage claimed is that judicious use of the second variant will reduce round-off error accrued in the first variant. Proofs and references are postponed to a second instalment, announced for the following issue of the same publication.

{While the first variant (which avoids having to compute the eigenrows of L) is missing from Duncan and Collar, its basic idea has been discussed elsewhere [see J. Wilkinson, Proc. Cambridge Philos. Soc. **50** (1954), 536–566, p. 548; MR **16**, 178]. The author also mentioned two references in a footnote added in proof. Feller and Forsythe [Quart. Appl. Math. **8** (1951), 325–331; MR **12**, 538] unify a wide class of such deflatior s in terms of simil rity transformations. The second variant seems to be new.}

G. E. Forsythe (Stanford, Calif.).

Bauer, F. L. Zur numerischen Behandlung von algebraischen Eigenwertproblemen höherer Ordnung. Z. Angew. Math. Mech. **36** (1956), 244–245.

This is a summary of a talk given at the Stuttgart meeting of the Gesellschaft für angewandte Mathematik und Mechanik, May 1956. The matrix eigenvalue prxblem

$$(\mathfrak{A}_0\lambda^n+\mathfrak{A}_1\lambda^{n-1}+\cdots+\mathfrak{A}_n\lambda^0)\mathfrak{C}=0$$

(where the $\mathfrak{A}_r$ are of order m and det $\mathfrak{A}_0\neq 0$) can be reduced to the ordinary eigenvalue problem for the matrix $\mathfrak{F}$ of order nm:

$$\mathfrak{F}=\begin{bmatrix} -\mathfrak{A}_0^{-1}\mathfrak{A}_1 & -\mathfrak{A}_0^{-1}\mathfrak{A}_2 & \cdots & -\mathfrak{A}_0^{-1}\mathfrak{A}_{n-1} & -\mathfrak{A}_0^{-1}\mathfrak{A}_n \\ E & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & E & 0 \end{bmatrix}$$

or for the matrix $\hat{\mathfrak{F}}$ obtained by reflecting $\mathfrak{F}$ in its northeast-southwest diagonal.

The author discusses the numbers of operations required to get the eigenvalues of $\mathfrak{F}$ or $\hat{\mathfrak{F}}$ by several known direct and iterative methods, as compared with the operations required for a general matrix of order nm. He advocates his "step-iteration" (Treppeniteration) for the simultaneous determination of several eigenvalues — a method to be described elsewhere. *G. E. Forsythe.*

Adachi, Ryuzo. On the Newton's method for the approximate solutions of simultaneous equations. Kumamoto J. Sci. Ser. A. **2** (1955), 259–272.

Newton's method of iteration for the solution of equations in one unknown is extended to systems of simultaneous equations. The paper derives formulas for equations in two unknowns but states that it is not difficult to extend the method to many unknowns.

If $f(x, y)=0$ and $\varphi(x, y)=0$ are the two simultaneous equations, and f and φ and their first and second partial derivatives are continuous, then the method in essence is:

1. Find two sets of values (α_1, β_1) and (α_2, β_2) such that

$$\varphi(\alpha_1, \beta_1)=0=\varphi(\alpha_2, \beta_2),$$
$$f(\alpha_1, \beta_1)>0, \quad f(\alpha_2, \beta_2)<0,$$

2. The first approximations (a, b) and $(\bar{a}, \bar{b})$ are taken as (α_1, β_1) or (α_2, β_2) depending on a lengthy condition involving first and second partial derivatives.

3. The second approximation is obtained from formulas:

$$a_1=a-\frac{f(a, b)\varphi_y(a, b)}{f_x(a, b)\varphi_y(a, b)-f_y(a, b)\varphi_x(a, b)}$$
$$b_1=\bar{b}-\frac{f(\bar{a},\bar{b})\varphi_x(\bar{a}, \bar{b})}{f_y(\bar{a}, \bar{b})\varphi_x(\bar{a}, \bar{b})-f_x(\bar{a}, \bar{b})\varphi_y(\bar{a}, \bar{b})}$$

and (a_1, b_1) is a better solution than either (α_1, β_1) or (α_2, β_2).

4. The procedure is reiterated using (a_1, b_1) and (α_1, β_1) or (α_2, β_2) depending on which satisfies step (1) and the author assures use of convergence.

The paper presents several numerical examples which pointedly illustrates that this method "can estimate not only the existence of the solutions but also the limits of the errors of the solution, but it has a defect in the point that the calculation is considerably troublesome."

S. Davis (Philadelphia, Pa.).

Morrison, D. R. A method for computing certain inverse functions. Math. Tables Aids Comput. **10** (1956), 202–208.

'; $f(x)$ is a continuous monotonic function for $0<x\leq a$ an it is possible to calculate $f(2x)$, $f(2x-a)$ for a given $f(x$ then a digit by digit process to evaluate $f^{-1}(y)$ as a bin y number is established.

T s process, although linear in convergence, may be rapid in use because the repeated loop often contains only a very few instructions. *D. C. Gilles.*

Muller, David E. A method for solving algebraic equations using an automatic computer. Math. Tables Aids Comput. **10** (1956), 208–215.

In determining the solutions of $f(x)=0$ ($f(x)$ a polynomial of degree n), the author proposes an iteration scheme to determine the next approximation x_{i+1} from three approximations x_i, x_{i-1}, x_{i-2} by using the inverse of a three point Lagrange formula. It is suggested that the roots be found in increasing order of magnitude, and that, each time a root is found, the degree of $f(x)$ be reduced by removing this root. The convergence of the method is established. *D. C. Gilles* (Manchester).

Mikeladze, Š. E. Approximate formulas for the multiple integral of a regular function. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 577–584. (Russian)

The point of departure is the reduction in the complex field of multiple integrals to single integrals by the formula:

$$(*)\quad \int_A^z\cdots\int_A^z w^{(n)}(z)dz^n=\{(n-1)!\}^{-1}\int_A^z(z-u)^{n-1}w^{(n)}(u)du.$$

The author then uses ideas developed in an earlier paper [same Soobšč. **17** (1956), 289–296; MR **18**, 479] to give methods of obtaining numerical integration formulas for the right side of (*), and hence for the left side of (*).

Formulas of both closed and open type are considered. In illustration the author gives numerical formulas for integrating $w'\equiv dw/dz=f(z,w)$, $w''=f(z,w,w')$, and $w''=f(z,w)$ over a network in a complex region.
G. E. Forsythe (Stanford, Calif.).

Gregory, R. T. A method for deriving numerical differentiation formulas. Amer. Math. Monthly **64** (1957), 79–82.

Using the Taylor expansion of $y(x)$ for a set of points x_j about x_i, a numerical differentiation formula for $y_i^{(k)}$ is obtained in terms of the ordinates y_j by elimination of the derivatives $y_i^{(s)}$, $s\neq k$, in the Taylor expansions. The error is discussed.
D. C. Gilles (Manchester).

Fisher, Michael E. Higher order differences in the analogue solution of partial differential equations. J. Assoc. Comput. Mach. 3 (1956), 325–347.

In a partial differential equation with two independent variables, derivatives in one variable can be replaced by differences so that the partial differential equation is replaced by a system of ordinary differential equations. In general the use of higher order differences introduces spurious solutions which may or may not damp out. If the differences are taken with respect to x, the author proposes using the equation to relate values of the function and its derivative at points intermediate to the x_i, and expressing these values in terms of the values at the x_i by means of interpolation formulas of suitable degree. This provides an extra degree of freedom which can be used to reduce by unity the number of spurious solutions. Several examples are discussed, and a table of interpolation coefficients for the functions and for derivatives up to order 3.
A. Householder (Madison, Wis.).

Malkevič, M. S. On the solution of integral equations in the theory of scattering of light in the atmosphere. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 1080–1090. (Russian)

In §1 the author considers the integral equation

$$(1)\qquad \varphi(\tau)=f(\tau)+\tfrac{1}{2}\int_0^{\tau^*}\varphi(t)E_1(|\tau-t|)dt,$$

arising the theory of the scattering of light in the atmosphere, when a parallel sheaf of rays enters the atmosphere at the top, the scattering being assumed isotropic and the ground non-reflecting. He discusses a process of numerical solution, in which (1) is integrated twice to obtain

$$(2)\qquad \tfrac{1}{2}\int_0^{\tau^*}\varphi(\tau)E_3(|\tau-t|)dt+C_1\tau+C_2+f_2(\tau)=0,$$

and $E_3(x)$ is then replaced in (2) by its best mean-square approximation of the form ae^{-bx} or, more generally, of the form

$$a_0e^{-b_0x}+\sum_{n=1}^{N}(a_n\sin b_nx+b_n\ \mathrm{ocs}\ b_nx).$$

A numerical example is given and, even with only a single term ae^{-bx}, is shown to give a better result than the Schwarzschild approximation.

In §2 the corresponding homogeneous equation

$$(3)\qquad \varphi(\tau)=\lambda\int_0^{\tau^*}\varphi(t)E_1(|\tau-t|)dt$$

is considered. Again a double integration is followed by the introduction of an approximation to the kernel; it turns out that the approximate characteristic values obtained for (3) in this way have a finite limit point. The author shows that by modifying the approximating kernel so as to drive this limit point to infinity, one obtains a much better set of solutions to (3), although the approximation to the kernel itself is worsened; the maximum error in the characteristic function of lowest order is reduced from more than 10% to lees than 2%.

In §3 it is shown that the same considerations are applicable to the equation

$$\psi(\tau)=\chi(\tau)+\tfrac{1}{4}g(\tau)\int_0^{\tau^*}\psi(\tau)[E_1(|\tau-t|)-E_3(|\tau-1|)]dt,$$

arising when the scattering is anisotropic and is given by a law of the form

$$\gamma(\tau;r,r')=1+g(\tau)\cos(r',r).$$

The case when $g(\tau)$ is constant is discussed in detail, with a numerical example, and some remarks are made about the case when $g(\tau)$ varies with the optical depth τ.
F. Smithies (Cambridge, England).

See also: Kostarčuk, p. 713; Slugin, p. 736; de Vito, p. 749; Tseng, p. 749; Bejar, p. 771; Kovalevsky, p. 782.

Graphical Methods, Nomography

Ulčar, Jože. Eine graphische Bestimmung von Gleichungslösungen. Bull. Soc. Math. Phys. Macédoine 6 (1955), 18–26. (Macedonian. German summary)

A method for graphical solution of equations of the type $F[f(x),g(x)]=0$ and of systems of equations $\Phi[f(x),g(y)]=0$, $F(x,y)=0$ is presented. The method is based on a very simple single-valued but not necessarily a one-to-one representation. It is very suitable in the case where the functions $f(x)$, $g(x)$ respectively $f(x)$, $g(y)$ and $F(x,y)$ are linear algebraic functions, and when they are monotone, although this condition is not necessary.

There are some misprints even in the titles of the paragraphs, but they are not misleading because the whole paper is elementary and easily readable.
T. P. Anđelić (Belgrade).

Slibar, A. Zur graphisch-numerischen Integration eines Simultansystems von gewöhnlichen, nichtlinearen Differentialgleichungen zweiter Ordnung. Österreich. Ing.-Arch. 10 (1956), 288–291.

See also: Baltaga, p. 726.

Machines and Modelling

See: Fisher, p. 767.

PROBABILITY

★**Richter, Hans. Wahrscheinlichkeitstheorie.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. xii+435 pp. DM 66.00.

This book illustrates the modern trend in advanced probability text-book writing, to include a self-sufficient treatment of the needed topics in measure theory. In Richter's book, the reader need not know any measure theory in advance, although he will have to read over 80 pages devoted to non-probabilistic measure theory before he gets beyond the most elementary probability concepts.

In spite of all this measure theory, however, and in spite of the now standard measure-theoretic definitions of probability concepts (a random variable is a measurable function, and so on) the author, presumably for pedagogical reasons, tries, both in his notation and discussion (and even in the index), to separate probability from measure theory. For example, in an early chapter he discusses various types of convergence of a sequence of measurable functions, and in a later chapter he discusses the various types of convergence of a sequence of random variables. In the first [second] treatment a function [random variable] is usually denoted by f or g [a], with a suitable subscript. In the second treatment, the new designation "strong convergence" is introduced for almost everywhere convergence, and the various types of convergence are defined independently of the first treatment. Then these types are translated back into those treated in measure language, each in a separate statement, and many theorems on sequences of measurable functions obtained in the first treatment are restated in probability language for sequences of random variables. Whether one regards this repetition as a valuable pedagogical device, as a method of stressing the identity between mathematical probability and measure theory, or as a disturbing hint that the two subjects are really different in some mysterious way, depends on the background and prejudices of the reader. At any rate the author's point of view has the advantage that he never simply flatly defines a probabilistic concept as a measure-theoretic one, but always leads up to the definition by a discussion in probability language.

The seven chapters cover the following material: I. Measure-theoretic foundations (measures and their construction); II. Concept of probability (intuitive and mathematical); III. Elements of probability theory (basic concepts and definitions); IV. Elements of integration theory; V. Random variables and general probability fields (independence, expectations, characteristic functions, convergence of sequences of distribution functions); VI. Special probability distributions (Gamma, multinomial, Gaussian and the distributions used in statistics derived from the Gaussian); VII. The convergence of random variables (sums of independent random variables, zero-one law, law of large numbers, central limit theorem).

The book's length, especially in view of the amount of explanatory discussion, does not allow the completeness of such a large and compactly written book as that by Loève [Probability theory, Van Nostrand, New York, 1955; MR **16**, 598] but what is treated is covered adequately and rigorously. The discursive style makes for easy reading, although it makes the book somewhat inconvenient for reference purposes. *J. L. Doob.*

★ **Mazurkiewicz, Stefan. Podstawy rachunku prawdopodobieństwa. [Foundations of the calculus of probability.] Prepared for print from the late author's manuscripts by Jerzy Łoś.** Państwowe Wydawnictwo Naukowe, Warszawa, 1956. iv+270 pp. zł. 27.

The aim of this book, as stated by the author, is to acquaint the reader with a modern foundation of the theory of probabilities and with those parts of the theory of functions of real variables which he considers indispensable in the study of probability. This is done in an unconventional manner, in a style combining remarkable clearness with rigor. The material is arranged in an Introduction, and Parts I and II. In the Introduction (pp. 1–22) the reader, who need not know mathematics beyond calculus, is supplied with the basic concepts of set theory and of set-theoretical topology in Cartesian spaces. — Part I, Elementary Probability Theory (pp. 23–174), is mainly devoted to an axiomatic foundation of probability theory, based on the observation that the objects to which probabilities are ascribed always form a Boolean algebra. Most of Part I is therefore a careful exposition of the theory of Boolean algebras. Less than forty pages deal with specific probability distributions and their properties, and cover the binomial distribution, Bernoulli's form of the law of large numbers, the de Moivre-Laplace theorem, the Poisson distribution, the multinomial distribution and its asymptotic form (χ^2-distribution). — Part II, Elements of the Theory of Real Functions (pp. 175–261), contains a presentation of topics indicated by the following chapter headings: Non-decreasing functions; Non-decreasing functions in n dimensions; Measures on Boolean algebras; Measures in Euclidean spaces; The Lebesgue-Stieltjes integral. *Z. W. Birnbaum* (Seattle, Wash.).

Weingarten, Harry. On the probability of large deviations for sums of bounded chance variables. Ann. Math. Statist. **27** (1956), 1170–1174.

The following two theorems are proved: I. Let $\{x_n, n=1, \cdots\}$ be a sequence of random variables with $-1 \leqq x_n \leqq a$, $a \leqq 1$. Suppose

$$E\{x_n | x_1, \cdots, x_{n-1}\} \leqq -u \max(|x_n| \, | x_1, \cdots, x_{n-1}),$$

where $0<u<1$. Then for any $t>0$

$$P\{x_1+\cdots+x_n \geqq t \text{ for some } n\} \leqq \theta^t,$$

where $\theta \neq 1$ is the positive root of

$$\frac{a+u}{a+1}\theta^{a+1}-\theta^a+\frac{1-u}{a+1}=0.$$

II. If $|x_n| \leqq 1$ and $E\{x_n | x_1, \cdots, x_{n-1}\}=0$ then for every $N>0$

$$P\left\{\frac{x_1+\cdots+x_n}{n} \geqq \varepsilon \text{ for some } n \geqq N\right\} \leqq 2[(1+\varepsilon)^{-\frac{1}{2}(1+\varepsilon)}(1-\varepsilon)^{-\frac{1}{2}(1-\varepsilon)}.]^N$$

Both results are improvements of theorems proved by Blackwell [same Ann. **25** (1954), 394–397; MR **15**, 882] and the techniques used are similar. *J. R. Blum.*

Dufresne, P. Problèmes de dépouillements. Publ. Inst. Statist. Univ. Paris **5** (1956), 75–89.

Continuation of the author's earlier work [Gaz. Mat., Lisboa **11** (1950), nos. 44–45, 8–14; MR **12**, 424]. Besides obtaining his previous formulas by another method, the author now evaluates the probability that a candidate will maintain a certain proportionate superiority in a vote count, over his opponent. There are no references to the voluminous literature on the problem. *J. L. Doob.*

Laha, R. G. On a characterization of the normal distribution from properties of suitable linear statistics. Ann. Math. Statist. **28** (1957), 126–139.

Let y and x be two random variables with finite variance. Denote by $E(y|x)$ the conditional expectation and by $V^2(y|x)$ the conditional variance of y, given x. The conditional distribution (c.d.) of y for fixed x is said to be L.R.H. (β, σ_0^2) [=author's abbreviation for linear regression and homoscedastic] if $E(y|x)=\beta x$ and also $V^2(y|x)=\sigma_0^2$, where β and σ_0^2 are constants. The author studies linear functions $X=\sum_{j=1}^n a_j x_j$, and $Y=\sum_{j=1}^n b_j y_j$ of independently, but not necessarily identically distributed two-dimensional random variables (x_j, y_j) such that

the c.d. of y_j for fixed x_j is L.R.H. (β_j, σ_{j0}^2) and derives a necessary and sufficient condition which must be satisfied in order that the c.d. of Y for fixed X be L.R.H. (β, σ_0^2).

This conditions leads to characterizations of the normal distribution. The proof uses an interesting extension of Cramér's theorem concerning the factorization of the normal distribution. This extension is due to A. A. Zinger and Yu. V. Linnik [Vestnik Leningrad. Univ. **10** (1955), no. 11, 51–56; Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 137–138; MR **17**, 753]. It is a valuable feature of the present paper that it contains a proof of Linnik's theorem and makes thus this result easily accessible. An interesting generalization of Linnik's theorem was given by D. Dugué [C. R. Acad. Sci. Paris **244** (1957), 715–717; MR **18**, 650]. *E. Lukacs* (Washington, D.C.).

Féron, R. Sur les distributions de probabilité de Laurent Schwartz et quelques unes de leurs applications au calcul des probabilités. Publ. Inst. Statist. Univ. Paris **5** (1956), 13–27.

The author discusses probability distributions from the point of view of the Schwartz theory, in which a probability distribution in n dimensions is identified with the corresponding Schwartz distribution (linear functional). As application, Fréchet's results on the upper and lower limits of the bivariate distribution functions having specified marginal distribution functions [Ann. Univ. Lyon. Sect. A. (3) **14** (1951), 53–77; MR **14**, 189] are derived and a vague statement by Bertaut [C. R. Acad. Sci. Paris **240** (1955), 152–154; MR **16**, 493] involving δ functions is made precise. *J. L. Doob* (Geneva).

Gervaise, Anne-Marie. Sur la somme de variables aléatoires positives. C. R. Acad. Sci. Paris **244** (1957), 840–842.

Suppose $F(x)$ is a continuous distribution function with $F(0)=0$. Let $X_1, \cdots, X_m$ be independent random variables with this distribution function. The distribution function of the random variable,

$$Y_m=(\sum X_i-\max(X_i))/(m-1),$$

is considered, and the following result is stated.

Suppose that for $0\leq y<1$ the equation

$$M(\beta)=\beta^{-1}\left(\int_0^\beta xdF(x)\right)\Big/\left(\int_0^\beta dF(x)\right)=y$$

admits a solution, $\beta=\beta(y)$. Let $k_m(u)$ be the solution of the equation $F(k_m(u))=1-u/m$, and let $u_0(y)$ be the solution of the equation, $\lim_{m\to\infty} M(k_m(u))=y$. Then Y_m has a limiting distribution as $m\to\infty$. When $M(k_m(u))$ is a constant c, which does not depend on u, the limiting distribution assigns probability one to the value c. When $M(k_m(u))$ varies with u, the limiting density function exists and is given by $e^{u_0(y)}$. *P. Meier.*

Chernoff, Herman; and Lieberman, Gerald J. The use of generalized probability paper for continuous distributions. Ann. Math. Statist. **27** (1956), 806–818.

The problem of plotting on probability paper is extended to continuous distributions which are completely specified except for scale and location parameters. Necessary and sufficient conditions are given to ensure that the plot which is optimal for estimating the scale parameter is also optimal for estimating each of the percentiles. (Authors' summary.) *H. Teicher.*

Kemp, C. D.; and Kemp, A. W. Generalized hypergeometric distributions. J. Roy. Statist. Soc. Ser. B. **18** (1956), 202–211.

Conditions on the parameters a, b and n are given under which the hypergeometric series $F(-a, -n, b-n+1; x)$ is a probability generating function, i.e., the coefficient of x^n may be interpreted as the probability that some non-negative integral-valued random variable assumes the value n. Urn model interpretations are provided for several sets of conditions, limiting distributions are discussed and the existence and nature of factorial moments are investigated. *H. Teicher.*

Lukacs, Eugene. On certain periodic characteristic functions. Compositio Math. **13** (1956), 76–80.

Let $F(x)$ be the cumulative distribution function of a probability distribution, and let

$$f(z)=\int_{-\infty}^{\infty} e^{izx}dF(x)$$

be its characteristic function (ch.f.). Let us say that $F(x)$ belongs to a lattice distribution when it is a step-function whose jumps are at some or all of the points of an arithmetic progression, and that $f(z)$ is an analytic ch.f. when it coincides with an analytic function in some neighbourhood of the origin. The author establishes propositions whose enunciations (as amended by him in a letter to the reviewer) are as follows. Theorem 1. An analytic ch.f. which is single-valued and periodic has either a real or a purely imaginary period. The period is real if, and only, if, the ch. f. belongs to a lattice distribution which has the origin as a lattice point. Corollary. A ch.f. which does not reduce to a constant cannot be doubly periodic with a period parallelogram contained in the interior of the strip of regularity. Theorem 2. A ch. f. is an entire periodic function (not $\equiv 1$) if, and only, if it is the ch.f. of a lattice distribution which has the origin as a lattice point.

H. P. Mulholland (Birmingham).

Kampé de Fériet, Joseph. Un problème de probabilité conditionnelle pour les fonctionnelles linéaires sur un espace de Banach. C. R. Acad. Sci. Paris **244** (1957), 24–27.

The author remarks that, if a Banach space has a probability measure with the property that every linear functional has a normal distribution with zero means, the conditional distribution of one linear functional, given a second, can be calculated as usual from a knowledge of the covariances. As an illustration he takes the space of continuous functions on the interval [0, 1], vanishing at the endpoints, with the uniform norm, and the probability measure of Brownian motion tied down at 0, 1. *J. L. Doob.*

Watson, G. S. Analysis of dispersion on a sphere. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. **7** (1956), 153–159 (1957).

Watson, G. S. A test for randomness of directions. Monthly Not. Astr. Soc. Geophys. Suppl. **7** (1956), 160–161 (1957).

The probability density on a sphere, $\exp(\kappa\cos\theta')$, where θ' is the angle between the polar and observation vectors, was suggested by R. A. Fischer [Proc. Roy. Soc. London. Ser. A. **217** (1953), 295–305; MR **15**, 139]. The first paper presents a series of tests of significance for hypotheses about the precision constant κ and the polar vector.

The second paper is concerned with a test for randomness of direction, for which $\kappa=0$. Let R be the vector resultant of N unit vectors. To define a significance test, a value R_0 is found such that, on the hypothesis of randomness, R exceeds R_0 with a specified probability α. The integral that gives this probability was given by Fischer (loc. cit.). Values of $\alpha=0.05$ and 0.01 are often used for this purpose. The appropriate value of R_0 is called a significance point. The author presents expansions that are useful in numerical applications, and gives a table of numerical values of 5 per cent and 1 per cent significance points for sample sizes $N=5$ to $N=20$. Directions are also given for obtaining the results for larger sample sizes. The study was made in connection with the analysis of paleomagnetic data. *D. Brouwer.*

Hamblen, John W. Distributions of roots of quadratic equations with random coefficients. Ann. Math. Statist. 27 (1956), 1136–1143.

The problem is to find the probability distribution of the roots η_1, η_2 of the quadratic equation $\eta^2-\xi_1\eta+\xi_2=0$ when ξ_1, ξ_2 are random real numbers with a given joint probability distribution of the absolutely continuous type with frequency function $f(\xi_1, \xi_2)$. Conditional frequency functions $g(\eta_1, \eta_2)$ and $h(\alpha, \beta)$ and found relative to the hypothesis $\xi_1^2-4\xi_2\geq 0$ and $\xi_1^2-4\xi_2\leq 0$, corresponding to the cases of real roots η_1, η_2 are complex conjugate $\eta=\alpha\pm i\beta$ respectively; and g, h are found explicitly in two special cases. *H. R. Pitt* (Nottingham).

Kimme, Ernest G. On the convergence of sequences of stochastic processes. Trans. Amer. Math. Soc. **84** (1957), 208–229.

For each positive integer n, let $\{x_{nk}, k\leq k_n\}$ be a finite sequence of mutually independent random variables, and, if $0\leq t\leq 1$, let $x_n(t)$ be the sum of the first $[tk_n]$ of these variables. Then $\{x_n(t), 0\leq t\leq 1\}$ is a stochastic process with independent increments. Applying limit theorems of Gnedenko, the author derives necessary and sufficient conditions that the finite-dimensional distributions of this process converge ($n\to\infty$), in various senses, to the corresponding ones of a limiting process with independent increments. The following theorem thus has as one application the calculation of distributions relating to continuous parameter processes from those for finite sums of random variables. [Cf. Erdös and Kac, Bull. Amer. Math. Soc. 52 (1946), 292–302 [MR 7, 459] for this type of study in the same spirit.] For each positive integer n let now $\{x_n(t), 0\leq t\leq 1\}$ be a stochastic process with independent increments, and let $\{x(t), 0\leq t\leq 1\}$ be another such process, with no fixed point of discontinuity. It is supposed that, for every finite set $(t_1, \cdots, t_m)$ in $[0, 1]$, the characteristic function of $[x_n(t_1), \cdots, x_n(t_m)]$ converges uniformly ($n\to\infty$), for each value of m, as $t_1, \cdots, t_m$ vary arbitrarily and the arguments of the characteristic functions vary in any finite interval, to the characteristic function of $[x(t_1), \cdots, x(t_m)]$. Let F be a functional defined, bounded, and uniformly continuous in the uniform topology, on the space D of real functions on $[0, 1]$, such that, if $f_n\in D$, and if the sequence $\{f_n, n\geq 1\}$ converges boundedly to f on $[0, 1]$ less a countable set, then $F(f_n)\to F(f)$. It is proved that then $E\{f[x_n(\cdot)]\}\to E\{f[x(\cdot)]\}$. This result extends one due to Donsker [Mem. Amer. Math. Soc. 6 (1951); MR 12, 723] who assumed that the $x(t)$ process was the Brownian motion (Wiener) process. For Prohorov's general approach to such problems, see Kolmogorov and Prohorov [Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, 1954, Deutscher Verlag der Wissenschaften, 1956, pp. 113–126; MR 18, 519]. *J. L. Doob* (Geneva).

Homma, Tsuruchiyo. On the theory of queues with some types of queue-discipline. Yokohama Math. J. 4 (1956), 55–64.

For the case of a single server, exponentially distributed times between arrivals, and arbitrary service times, the author gives conditions for the Markov chain of queue lengths at conclusions of service periods to be ergodic, null-recurrent, or transient, under each of the following two modifications of the usual assumptions: (1) as the service of the ith customer be gins, r_i-customers leave the queue, the r_i being truncated (at the queue length) from independent, bounded, identically distributed random variables; (2) the arrival times are instants when groups of customers arrive, the numbers of customers in various groups being independent and identically distributed with arbitrary distribution. *J. Kiefer.*

Udagawa, Kanehisa; and Nakamura, Gisaku. On a certain queuing system. Kōdai Math. Sem. Rep. **8** (1956), 117–124.

"Throughout this paper, we will assume that (1) the interarrival times of successive customers are distributed by a negative exponential law with a parameter $1/\lambda$, (2) the first join will be first served and (3) the distribution of service times is also a negative exponential law. Considering discrete time points (epochs) at which services start, the number of customers at successive epochs constitutes a simple Markov chain and the queue becomes a birth process in any interval between two successive epochs. In this paper, first, the transition probability of the simple Markov chain will be calculated and next some characteristics of the birth process will be investigated and further discussions will be done". (From authors' summary). *J. Wolfowitz.*

Lévy, Paul. Random functions: a Laplacian random function depending on a point of Hilbert space. Univ. California Publ. Statist. 2 (1956), 195–205.

The author considers a random variable $X(A)$, depending on the point A of a Hilbert space, under the hypothesis that the $X(A)$ stochastic process is Laplacian and that $X(B)-X(A)$ has zero expectation and variance the distance between A and B. The problem is the analysis of the deterministic character of this stochastic process. Further details and extensions of earlier results are given [C. R. Acad. Sci. Paris 239 (1954), 1181–1183, 1584–1585; **240** (1955), 1043–1044; MR 16, 495, 939]. For example, if $X(A)$ is known in a certain sense on a closed "surface", then it is determined inside the surface. If the surface is a sphere it is sufficient to know $X(A)$ on a "median curve" (one that divides the surface into two parts of equal area in an appropriate sense). *J. L. Doob* (Geneva).

Talacko, Joseph. Perk's distributions and their role in the theory of Wiener's stochastic variables. Trabajos Estadist. 7 (1956), 159–174. (English. Spanish summary)

In J. Inst. Actuar. 63 (1932), 12–57, W. Perks introduced a family of distribution functions suitable for graduation of mortality statistics. In the present, very interesting, paper the properties of Perks' distributions are studied. Their connection with the Brownian movement is shown. *P. Johansen* (Copenhagen).

Wigner, Eugene P. Characteristic vectors of bordered matrices with infinite dimensions. II. Ann. of Math. (2) 65 (1957), 203–207.

The basic integral equation of part I [Ann. of Math. (2) 62 (1955), 548–564; MR 17, 1097] is cast into simpler form, from which asymptotic formulae are now deduced more easily.
J. G. Wendel (Ann Arbor, Mich.).

Steinhaus, H. Calculus of probabilities as a research tool in natural sciences and engineering. Prace Mat. 2 (1956), 27–55. (Polish)

A survey of modern developments in probability theory and mathematical statistics, with particular reference to achievements and endeavours of Polish mathematicians and statisticians. *S. K. Zaremba* (Wolverhampton).

See also: Finch, p. 711; Dupač, p. 721; Carlson, p. 771; Murty, p. 772; Dvoretzky, Kiefer and Wolfowitz, p. 772.

STATISTICS

★**van der Waerden, B. L. Mathematische Statistik.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. ix+360 pp. DM 46.00.

This book is intended as an introductory text in statistics for students with considerable mathematical background, including Lebesgue integration, matrix theory, and some knowledge of complex function theory. The book, however, gives the impression of a collection of fairly well-written chapters, in each of which the author is more or less unaware of what has already been done in the book, and in addition, the author seems unwilling to make use of the mathematics he has assumed. Occasionally, however, fairly deep results are used. Proofs of important theorems which can easily be carried out with the mathematics assumed are frequently omitted, for example, that of the asymptotic distribution of maximum likelihood estimates, pp. 180–181, although only a few more steps would be needed to obtain a rigorous treatment; also more of the proofs of limit theorems could easily be given.

The chapters are: general foundations; probability and frequency; mathematical tools; empirical determination of distribution, mean, and variance; Fourier integrals and limit theorems; Gaussian variables and Student's test; least squares; estimation; evaluation of observed frequencies; bio-assay; tests of hypotheses; rank tests; and correlation. A great part of the text is devoted to examples and a collection of tables is included. However, the theory is inadequately covered so that the text is decidedly inferior to that of Mood [Introduction to the theory of statistics, McGraw-Hill, New York, 1950; MR 11, 445] and Schmetterer [see MR 18, 681].

Therefore, the reviewer feels that despite its clarity and teachability, this book does not succeed in its purpose.
H. Rubin.

Steyn, H. S. On regression properties of discrete systems of probability functions. Nederl. Akad. Wetensch. Proc. Ser. A. 60=Indag. Math. 19 (1957), 119–127.

The multivariable probability function $f(x_1, x_2, \cdots, x_n)$ is shown to have a regression function, of x_1 on all remaining variables, of form

$$Q(x_2, x_3, \cdots, x_n)/R(x_2, x_3, \cdots, x_n)$$

with Q and R polynomials if and only if

$$[\theta_1 R(\theta_2, \cdots, \theta_n)M(\alpha_1, \cdots, \alpha_n)] = [Q(\theta_2, \cdots \theta_n)M(\alpha_1, \cdots, \alpha_n)]$$

with $M(\alpha_1, \cdots, \alpha_n)$ the moment generating function, $\theta_i = \partial/\partial\alpha_i$, and the brackets indicating final evaluation at $\alpha_1 = 0$. This result is applied to multinomial Bernoulli, Pascal and hypergeometric distributions, getting linear regressions, and to other bivariate functions of hypergeometric type for nonlinear regressions. *J. Riordan.*

Bejar, Juan. Median regression and linear programming. Trabajos Estadist. 7 (1956), 141–158. (Spanish. English summary)

The median regression curve $y=g(x)$ is defined by $\int_{-\infty}^{g(x)} f(y|x)dy = \frac{1}{2}$, where $f(y|x)$ is the conditional probability density function of y for given x. This definition of $g(x)$ is that which minimizes $E|y-g(x)|$. The author considers certain properties of the two distributions of x obtained by integrating y over all values $\lessgtr g(x)$, respectively, and in particular when $g(x)$ is a polynomial in x. He also discusses the empirical fitting of $y=\alpha+\beta x$ by minimizing $\sum |y-\alpha-\beta x|$ and draws an analogy between this problem and that of linear programming. *H. L. Seal.*

Ghurye, S. G. Note on asymptotic estimation of parameters of an autoregressive process. Ganita 6 (1955), 1–7 (1956).

The method of instrumental variables is adapted for estimating an autoregressive process with serially correlated disturbances. Partial proofs of consistency and normal convergence. *H. Wold* (Uppsala).

Carlson, Phillip G. A least squares interpretation of the bivariate line of organic correlation. Skand. Aktuarietidskr. 39 (1956), 7–10.

Let X, Y be a pair of random variables whose means m_x, m_y, variances σ_x^2, σ_y^2, and covariance $\rho\sigma_x\sigma_y$ are known. The "line of organic correlation" is the straight line through (m_x, m_y) whose slope α^* is numerically equal to σ_y/σ_x and has the same sign as ρ. Certain uniqueness properties of this line were established by W. H. Kruskal [Biometrics 9 (1953), 47–58; MR 14, 890]. The author shows that there is a unique direction such that the least squares principle applied in that direction yields a summary line whose parameters are independent of ρ except for the sign of the slope. In fact, the expectation of the square of the distance of (X, Y) from a line, if reckoned in the direction with slope $-\alpha^*$, attains its minimum when the line coincides with that defined above.
H. P. Mulholland (Birmingham).

Des Raj. A note on the determination of optimum probabilities in sampling without replacement. Sankhyā 17 (1956), 197–200.

Suppose that the probabilities with which the several members of a finite population are selected for a sample are allowed to differ and to depend on what items have already been drawn, subject to the condition that the probability that any two items appear in the sample together is predetermined. Horvitz and Thompson discussed this problem at length [J. Amer. Statist. Assoc. 47 (1952), 663–685; MR 14, 777]. The present author finds how to assign these probabilities so as to minimize the variance of the unbiased estimator of the population

total, in the special case where the sample size is two and the variate of interest is a linear function of another variate which is known. *S. W. Nash* (Vancouver, B.C.).

Kraft, C.; and LeCam, L. A remark on the roots of the maximum likelihood equation. Ann. Math. Statist. 27 (1956), 1174–1177.

Examples are given to show that the maximum likelihood estimator either need not exist or may not be a consistent estimator (of the parameter being estimated), even though a root of the likelihood equation is a consistent estimator. *J. Wolfowitz* (Ithaca, N.Y.).

Geidel, Hans. Zur Anwendung von Gleitmittelwertverfahren bei der Auswertung von Feldversuchen. Mitt. Math. Sem. Giessen Beiheft 2 (1956), 86 pp.

Moore, P. G. The estimation of the mean of a censored normal distribution by ordered variables. Biometrika 43 (1956), 482–485.

Let $x_1, x_2, \cdots, x_n$ be an ordered sample from a normal distribution. The author considers unbiased estimates of the mean, based on some values of this sample, e.g. $(x_r+x_{n+1-r})/2$, or the properly weighted mean of x_r and x_s where r and s are not symmetric subscripts, or a weighted mean of $x_r, x_{n+1-r}, x_s, x_{n+1-s}$. Using approximate expansions given by David and Johnson [Biometrika 41 (1954), 228–240; MR 16, 382] and in a "late note", results by Teichroew [Ann. Math. Statist. 27 (1956), 410–426; MR 18, 238], the author computes the efficiencies of these estimates and compares them with the efficiencies of maximum likelihood estimates found by Gupta [Biometrika 39 (1952), 260–273; MR 14, 487]. *Z. W. Birnbaum* (Seattle, Wash.).

van Eeden, Constance. Maximum likelihood estimation of ordered probabilities. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 444–455.

This is identical with Math. Centrum Amsterdam. Statist. Afdeling Rep. **S 196 (VP7)** (1956) [MR 17, 982]. *B. Epstein* (Detroit, Mich.).

Roy, S. N. A note on "Some further results in simultaneous confidence interval estimation". Ann. Math. Statist. 27 (1956), 856–858.

Let S_1 and S_2 stand for the dispersion matrices of random samples drawn from independent p-variate normal populations with dispersion matrices Σ_1 and Σ_2 respectively; let $c(M)$ denote a characteristic value of matrix M; and let $c_{1\alpha}$, $c_{2\alpha}$ be such constants that

$$P\{c_{1\alpha}\leq \text{all } c(S_1S_2^{-1})\leq c_{2\alpha}|\ \Sigma_1=\Sigma_2\}=1-\alpha.$$

Then, with confidence $\geq 1-\alpha$,

$$c_{\max}(S_1S_2^{-1})/c_{1\alpha}\geq \text{all } c(\Sigma_1\Sigma_2^{-1})\geq c_{\min}(S_1S_2^{-1})/c_{2\alpha},$$

even when $\Sigma_1\neq\Sigma_2$. This improves a previous result [same Ann. 25 (1954), 752–761; MR 16, 382]. *S. W. Nash.*

Murty, V. N. A note on Bhattacharyya bounds for the negative binomial distribution. Ann. Math. Statist. 27 (1956), 1182–1183.

The author calculates the Bhattacharyya lower bounds for the variance of an unbiased estimate of p for the negative binomial distribution, and shows that with increasing order they tend to the sharp lower bound. *E. L. Lehmann* (Berkeley, Calif.).

Kudô, Akio. On the confidence interval of the extreme value of a second sample from a normal universe. Bull. Math. Statist. 6 (1956), 51–56.

The paper discusses the problem of making inferences about the extreme value of a second independent sample $y_{\max}=\max(y_1, y_2, \cdots, y_m)$ under the situation where we already know the sample mean $\bar{x}$ and the sample variance s^2 from a first independent sample of size n. The formulation follows that of the two sample theory due to Kitagawa [Mem. Fac. Sci. Kyûsyû Univ. A. 5 (1950), 139–180; MR 13, 854], while the approximation is given along the lines of the method developed in another paper of the author, which yields

$$P(y_{\max}<\bar{x}+\lambda s)\cong \sum_{k=0}^{\infty}\frac{(2k)!}{k!}(\tfrac{1}{2}+\lambda^2(1-a(n))^2)^k A_{m,2k}(a(n)\lambda),$$

where $a_n=(2\pi^{-1})^{\frac{1}{2}}\,\Gamma(\frac{1}{2}n)/\Gamma(\frac{1}{2}(n-1))$ and $|A_{n,k}(x)|$ are defined by

$$A_{n,k}(x)=\sum_{\nu_1+\cdots+\nu_n=k}\frac{1}{\nu_1!}\phi^{(\nu_1)}(x)\frac{1}{\nu_2!}\phi^{(\nu_2)}(x)\cdots\frac{1}{\nu_n!}\phi^{(\nu_n)}(x)= \sum_{\nu=0}^{k}\frac{1}{\nu!}\phi^{(\nu)}(x)A_{n-1,k-\nu}(x),$$

where $\phi^{(\nu)}(x)$ is the νth derivative of the standardized normal distribution function.

The paper also suggests another approximation which will be useful for practical purposes when m becomes large. *T. Kitagawa* (Fukuoka).

Washio, Yasutoshi; Morimoto, Haruki; and Ikeda, Nobuyuki. Unbiased estimation based on sufficient statistics. Bull. Math. Statist. 6 (1956), 69–93.

The paper gives the unique unbiassed sufficient estimate $\delta(u)$ for a given function $\theta(\tau)$ of the parameter τ under the condition that there exists a sufficient statistic u for τ whose distribution is given by the exponential type $A(\tau)e^{-\tau u+v(u)}$. The solution is given by $\delta(u)=e^{-v(u)}w(u)$, where

$$w(u)=\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{\theta(z)}{A(z)}e^{zu}du$$

under a fairly general condition, and its essential aspect can also be interpreted as how to find the linear translatable functional operator Λ in the sense of Kitagawa such that $\Lambda_u e^{\tau u}=\theta(\tau)e^{\tau u}$. The paper gives various concrete examples when $\theta(\tau)=\tau^k$, $e^{c\tau}(2\pi)^{-1}\exp(-\frac{1}{2}(y-\tau)^2)$ (for a fixed y) etc. concerning normal, chi-square distributions as well as discrete distributions such as Poisson and binomial ones, from the point of view of the respective transformations which secure the unified aspect of the approach. The paper discusses the variances of the estimates and also two unknown parameters. As its special cases the paper includes some results given by Rao [Proc. Cambridge Philos. Soc. 45 (1949), 213–218; MR 10, 466] and Kolmogoroff [Izv. Akad. Nauk SSSR. Ser. Mat. 14 (1950), 303–326; MR 12, 116; 15, 452] concerning the unbiassed estimates of a few functions of unknown parameters. *T. Kitagawa* (Fukuoka).

Dvoretzky, A.; Kiefer, J.; and Wolfowitz, J. Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. Ann. Math. Statist. 27 (1956), 642–669.

Suppose $X_1, X_2, \cdots, X_n$ are independent, identically distributed chance variables, with unknown cumulative

distribution function $F(x)$. The problem is to estimate $F(x)$. Let ϕ_n be a generic symbol for a decision procedure, and let D_n denote the class of all well-defined decision procedures. $S_n(x)$ denotes the sample cumulative distribution function based on $X_1, \cdots, X_n$: that is, $S_n(x) =$ (number of $X_i \leq x)/n$. ϕ_n^* denotes the decision procedure which chooses $S_n(x)$ as the estimate of $F(x)$. For any estimate $Q(x)$, let $r(Q, F, x)$ denote $Q(x)-F(x)$, and $r(Q, F)$ denote $\sup_x|r(Q, F, x)|$. Let $E(L; F; \phi_n)$ denote the expected loss when the loss function is L, the distribution function is F, and the decision procedure ϕ_n is used. The authors show that if the loss function L is $W(n^{\frac{1}{2}}r(Q, F))$, where $W(r)$ is any positive and non-decreasing function of r satisfying certain mild conditions, then

$$\lim_{n\to\infty} \frac{\sup_F E(L; F; \phi_n^*)}{\inf_{\phi_n \in D_n} \sup_F E(L; F; \phi_n)} = 1.$$

The same result holds if the loss function L is

$$\int_{-\infty}^{\infty} V(n^{\frac{1}{2}}r(Q, F, x), F(x))dF(x),$$

where $V(y, z)$ is non-negative, symmetric in y, and non-decreasing in y for $y \geq 0$. Similar results hold for the problem of estimating the parameters of a multinomial distribution. *L. Weiss* (Eugene, Ore.).

Ghosh, M. N. Strong convergence of Robbins and Monro and Kiefer and Wolfowitz processes. Bull. Calcutta Math. Soc. **48** (1956), 25–32.

Under regularity conditions similar to those of Blum [Ann. Math. Statist. **25** (1954), 382–386; MR **15**, 973], the author proves the results stated in the title; these proofs were apparently obtained independently by him in 1954. *J. Kiefer* (Ithaca, N.Y.).

Weiss, Lionel. A certain class of tests of fit. Ann. Math. Statist. **27** (1956), 1165–1170.

Let $X_1, \cdots, X_n$ be independently and identically distributed on the unit interval, let $Y_1 \leq Y_2 \leq \cdots \leq Y_n$ be the ordered X_i's, and let $W_j = Y_j - Y_{j-1}$ (with $W_1 = Y_1$, $W_{n+1} = 1 - Y_n$). The author considers the critical regions $\sum W_j^u > K_n$ where $u > 1$, for testing whether or not the X_i's are uniformly distributed, and proves consistency against a wide class of alternatives. Among tests based on the ordered W_j's, that described above with $u=2$ is unbiased of Type A. *J. Kiefer* (Ithaca, N.Y.).

Adhikari, Bishwanath Prosad. Analyse discriminante des mesures de probabilité sur un espace abstrait. C. R. Acad. Sci. Paris **244** (1957), 845–846.

(1) If $\mu_1, \mu_2, \cdots, \mu_k$ are k simple hypotheses, a set of k acceptance regions $R_1, R_2, \cdots, R_k$ is determined with the property that the k probabilities of rejecting μ_i when μ_i is true, $i=1, 2, \cdots, k$, are equal and this common value is a minimum. (2) If there exist a priori probabilities for the μ_i, the R_i are determined so that the probability of a wrong decision is a minimum. (3) A certain "generalized error" in discriminating between μ_1 and μ_2 is stated not to decrease under a measurable transformation of the sample space. The paper contains no proofs.
D. A. Darling (Ann Arbor, Mich.).

★**Fraser, D. A. S. Nonparametric methods in statistics.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1957. x+299 pp. $8.50.

Although the field of nonparametric statistics has grown rapidly in recent years, no book devoted to this subject as such has been available so far. The present book is "an attempt to collect and unify these diverse developments". It is intended as a second course in mathematical statistics. "Prerequisites are a knowledge of calculus and familiarity with an introduction to statistics as is found in Hoel — Introduction to Mathematical Statistics [John Wiley and Sons, New York, 2nd ed. 1954]. A knowledge of measure theory is not necessary, the essential ideas of measure being introduced with a statistical interpretation." Actually, the outline of measure and integration on the first 16 pages is hardly sufficient to make the reader familiar with theorems like those of Fubini and Radon-Nikodym which are repeatedly used in the book.

The first two chapters (pp. 1–124) contain a survey of probability concepts and of general statistical inference. The remaining five chapters deal with nonparametric statistics and have the following titles. 3. Nonparametric problems. 4. The estimation of real parameters and tolerance regions. 5. The theory of hypothesis testing. 6. Limiting distributions. 7. Large sample properties of tests. Each chapter is followed by a collection of problems for solution and a list of references.

The book emphasizes methods for deriving statistical procedures which are optimal in some sense, or at least have some (frequently) desirable properties like unbiasedness. This gives the book a systematic character, and it succeeds well in the presentation of these methods. On the other hand, certain problems for which no "optimal" solutions are known have been neglected even where intuitively appealing methods are available. In particular, no tests of fit or confidence belts for a distribution function are discussed (even though the chi square test of fit is mentioned as one of the oldest nonparametric methods) and neither empirical distribution functions nor the statistics of Kolmogorov, Smirnov, and von Mises are mentioned (although the distance functions which underly these statistics appear in a discussion of "the problem of fit").

Another characteristic feature of the book is that whereas asymptotic distributions are given much (and deserved) attention, no exact distributions of rank statistics in the case of equally probable rank permutation are considered, with one single exception in section 6.7. The book contains no tables of distributions.

The more important statistical theorems and some of the less well-known probability theorems are proved in a clear and rigorous way, while for proofs of other theorems references are given. A regrettable exception is the statement without proof of a lemma attributed to Hunt and Stein (p. 106) which has been quoted in the literature for years but a proof of which has never been published. A new, simplified proof of the author's theorem on tolerance regions is given in section 4.3. A hitherto unpublished result of the reviewer on the asymptotic power of a two sample rank test proposed by Fisher and Yates is contained in section 7.5. *W. Hoeffding.*

van Eeden, Constance. Maximum likelihood estimation of partially or completely ordered parameters. I. Nederl. Akad. Wetensch. Proc. Ser. A. **60** = Indag. Math. **19** (1957), 128–136.

This paper treats the problem of finding maximum likelihood estimates of a finite partially or completely ordered set of parameters of probability distributions. A special case of this problem, that of finding the maximum likelihood estimates of a finite ordered set of probabilities,

was treated by the author in two earlier papers [van Eeden, Math. Centrum Amsterdam. Statist. Afdeling Rep. **S 188** (**VP5**) (1956); **S 196** (**VP6**) (1956); MR **17**, 640; 982]. A similar problem was treated by Ayer, Brunk, Ewing, Reid, and Silverman [Ann. Math. Statist. **26** (1955), 641–647; MR **17**, 504]. *B. Epstein.*

Kamat, A. R. A two-sample distribution-free test. Biometrika 43 (1956), 377–387. Addendum by D. E. Barton 386–387.

Let $x_1, \cdots, x_n$ and $y_1, \cdots, y_m$, $m \geqq n$, be random samples. The hypothesis H_0 that the distributions of x and y are equivalent is tested against the alternative hypothesis of equal locations but different spreads. The test statistic proposed is $D_{n,m}=R_n-R_m+m$, where R_n and R_m are the ranges of the ranks of x and y, respectively, after pooling the samples and arranging them in order of magnitude. H_0 is rejected for large or small values of $D_{n,m}$. The exact distribution of $D_{n,m}$ is derived, and the upper and lower .5, 1.0, 2.5 and 5.0 percentage points are given for $n=2(1)10$, and $m+n=7(1)20$. The test is an alternative to a test suggested by Rosenbaum [Ann. Math. Statist. **24** (1953), 663–668; MR **15**, 450]. In an addendum D. E. Barton has derived limiting distributions of $D_{n,m}$ for $m \to \infty$ and $n \to \infty$ or n finite.

D. M. Sandelius (Göteborg).

Finney, D. J.; and Cope, F. W. The statistical analysis of a complex experiment involving unintentional constraints. Biometrics 12 (1956), 345–368.

This paper discusses the analysis of yields from a complex factorial experiment on cacao in Trinidad. Originally conceived as a plaid square for a $2^3 \times 4$ set of treatments, circumstances forced a change to partial confounding in randomized blocks for a 2×4^2 set. The change was made after certain factors had been introduced, and therefore some constrains from the earlier design remained. Yields from the experiment in 1954–55 are used for illustration of the methods for analysis and summarizing the data neglecting the additional constraints and a multiple regression technique is described to take account of those constraints. *Om P. Aggarwal.*

Cox, D. R. A note on weighted randomization. Ann. Math. Statist. 27 (1956), 1144–1151.

In simple statistical designs the principle of randomization implies the selection with equal probability of a single arrangement from the totality of admissible arrangements. This note shows that in designs wherein a concomitant variable is known prior to allocation of treatments to the units, an unbiased between treatment mean square can be produced by weighted randomization, i.e. assigning to each arrangement of the set a specified weight, R_{xx}. Let $x_1, \cdots, x_N$ be a set of fixed constants associated with the i units, $i=1, \cdots, n$; and let there be n repetitions of τ treatments, $n\tau=N$. For any arrangement let

$$\bar{x}_\mu=n^{-1}\textstyle\sum_\mu x_i$$

be the mean of the concomitant variable associated with the μth treatment. Then

$$R_{xx}=\sum_{\mu=1}^{\tau}\sum_\mu(x_i-\bar{x}_\mu)^2.$$

M. E. Terry (Murray Hill, N.J.).

Bliss, C. I.; Greenwood, Mary L.; and White, Edna Sakamoto. A rankit analysis of paired comparisons for measuring the effect of sprays on flavor. Biometrics 12 (1956), 381–403.

This is an expository paper giving a thorough analysis of a paired comparisons experiment (tasting applesauce). The principal method used is analysis of variance for paired comparisons along the lines proposed by H. Scheffé [J. Amer. Statist. Assoc. **47** (1952), 381–400; MR **14**, 488], and replacing degree of preference by the corresponding rankit. Finally there is a comparison with other methods of analysis, which have appeared mainly in Biometrics. *S. W. Nash* (Vancouver, B.C.).

Des Raj. On sampling with probabilities proportionate to size. Ganita 5 (1954), 175–182 (1955).

For a finite population, units are to be sampled with probabilities proportional to the size of on auxiliary variate which is known for each unit. This paper studies conditions under which the variance of the estimated mean of the character under consideration is smaller than the variance of the corresponding estimator gotten using equal probability sampling. It is possible to have perfect correlation between the two variates and still have the equal probability method give the smaller variance. If there is a linear regression of the new variate on the auxiliary variate, proportional probability sampling is at its best when the regression line goes through the origin. *S. W. Nash* (Vancouver, B.C.).

Gini, Corrado. Généralisations et applications de la théorie de la dispersion. Metron 18 (1956), no. 1-2, 1–75.

This investigation, begun more than forty years ago and carried out in the tradition of Lexis, presents indices of dispersion suitable for all kinds of observations. Urn schemes are considered for each modification of the sampling scheme. In most cases the index squared is the ratio of the between groups mean square divided by the mean square for total variation. Formulas (2, 1) and (3,1) give quantities whose squares multiplied by the appropriate number of degrees of freedom equal the binomial and Poisson indices of dispersion respectively. There are many examples considered in a descriptive vein, but no distribution or probability statements are made about any of the many forms of the index of dispersion given. *S. W. Nash* (Vancouver, B.C.).

Ramakrishnan, Alladi. A physical approach to stochastic processes. Proc. Indian Acad. Sci. Sect. A. **54** (1956), 428–444.

The paper contains an expository account of the mathematical fomulation of stochastic processes of physical interest. *R. Bellman.*

See also: Bertaut, p. 717; Pollak, p. 722; Tseng, p. 749; Steinhaus, p. 771; Bastin and Kilmister, p. 782.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Sbrana, Francesco. Su una questione di statica. Boll. Un. Mat. Ital. (3) 11 (1956), 588–590.

This note gives some comments and explanatory remarks concerning two earlier papers by the author [same Boll. (3) 8 (1953), 123–127; 11 (1956), 123–125; MR 15, 68; 18, 79]. *L. A. MacColl.*

Unthank, H. W. Sphere rolling on a cone. Math. Gaz. 41 (1957), 53–54.

A uniform sphere rolling without slipping on the inner surface of a circular cone whose axis is vertical has a constant vertical resolute of angular velocity.... I do not know of any proof simpler than the following. *From the introduction.*

Kobrinskiĭ, A. E. Some questions of the dynamics of mechanisms with elastic connections. III. Trudy Inst. Mašinoved. 16 (1956), no. 61, 23–50 (2 plates). (Russian)

Parts I and II are listed in MR 16, 295.

Hölder, Ernst. Die Dynamik des starren Körpers in einem nichteuklidischen Raum. Abh. Math. Sem. Univ. Hamburg 20 (1956), 242–252.

This paper, dedicated to the seventieth birthday of Professor W. Blaschke, extends the validity of the dynamical differential equations, derived by the author in a previous paper [Z. Angew. Math. Mech. 19 (1939), 166–176], to the motion of rigid bodies in a non-euclidean elliptic and hyperbolic space. At the same time it is a complement in a way to some results obtained by Blaschke [Nichteuklidische Geometrie und Mechanik, 1943].

The author starts, firstly, from the relations which exist between the absolute derivatives of the impulses and the antisymmetric helicoidal velocity-tensor of the second order π_{ij} $(i,j=0, 1, 2, 3)$, and, secondly, from the dynamical equations he has derived for the motion of rigid bodies in an euclidean three dimensional space containing still another antisymmetric tensor whose coordinates are forces and the moments of forces. He adapts these equations to elliptic and hyperbolic spaces introducing suitable metrics, and then derives from Hamilton's principle the dynamical equations in Lagrangian form. He also gives Lagrangian equations expressed explicitly by means of the coordinates of the helicoidal velocity-tensor with respect both to a fixed and a moving system of reference. *T. P. Anđelić.*

Petrašen', G. I.; Marčuk, G. I.; and Ogurcov, K. I. On Lamb's problem for a half-space. Leningrad. Gos. Univ. Uč. Zap. 135. Ser. Mat. Nauk 21 (1950), 71–118. (Russian)

The "method of an incomplete separation of variables" seems to be appropriate for the solution of the Lamb's problem and related problems. In the two-dimensional case of a pulse due to stresses concentrated in a part of the boundary the solution is built up from double integrals and the relationship of this method to the method of complex solutions is shown. The use is made of Fourier and Mellin integrals and Laplace transforms but the single quoted investigation from the world literature is that of Rayleigh [Proc. London Math. Soc. 17 (1885), 4–11]. *W. Jardetzky* (New York).

Stojanovitch, Rastko. Brachistochronic motion of non-conservative dynamical systems. Tensor (N.S.) 6 (1956), 104–107.

The brachistochronic motion between two fixed configurations of a dynamical system is here discussed in the case where the field of force is non-conservative, and it it shown that the motion is determined by a system of differential equations of the third order. *A. J. McConnell* (Dublin).

Liu, Vi-Cheng. On the motion of a projectile in the atmosphere. Z. Angew. Math. Phys. 8 (1957), 76–82.

The equation of rectilinear motion of a projectile which moves in an atmosphere, of which the density decreases exponentially with altitude, is solved. It is found that the velocity of the projectile can be expressed explicitly in terms of confluent hypergeometric functions. This theory is applied to treat two specific problems: (1) the flight analysis of a sounding rocket during the free-flight period and (2) the calculation of ambient temperature from the trajectory of a spherical projectile. (From the author's summary.) *E. Leimanis.*

See also: Singer, p. 782.

Statistical Mechanics

Rytov, S. M. On thermal fluctuations in distributed systems. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 371–374. (Russian)

A theory of fluctuations is dissipative systems has been developed by Callen and others for the case of a finite number of thermodynamic parameters [R. F. Greene and H. B. Callen, Phys. Rev. (2) 88 (1952), 1387–1391; MR 16, 204], as a generalization of Nyquist's theory of voltage fluctuations in an electrical conductor. The author gives a formal extension to continuously distributed systems. The forcing and response functions are represented first by Fourier integrals over the time variable and the correlation matrices of the resulting Fourier components, which are functions of the space coordinates, are expressed in terms of the delta-function and its derivatives. The formalism is then extended by introducing Fourier integrals over both space and time coordinates. {The treatment is purely formal, no consideration being given to measure and existence problems associated with the transition to continuously distributed systems.}

E. L. Hill (Minneapolis, Minn.).

Lewis, M. B. Relation between the canonical and grand canonical ensemble. Phys. Rev. (2) 105 (1957), 348–353.

This paper settles two questions which had been left unanswered in previous mathematical discussions of the existence and general properties of the pressure in the canonical and grand canonical ensemble theories. Whereas the pressure in the latter theory has been studied by Yang and Lee [Phys. Rev. (2) 87 (1952), 404–409; MR 14, 711], Van Hove derived its main properties in the ca-

nonical theory [Physica 15 (1949), 951–961]. The present paper establishes that the grand canonical pressure is identical with the canonical pressure whenever the latter exists. In addition the average density in the grand ensemble is shown to exist except in the case where the canonical theory gives condensation (i.e. constant pressure over a whole interval of densities). The treatment holds for classical systems, the forces having a finite range and hard cores of non-vanishing radius.

L. Van Hove (Utrecht).

Belyaev, S. T.; and Budker, G. I. Relativistic kinetic equation. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 807–810. (Russian)

A relativistically invariant distribution function is defined and its relation to the usual distribution function determined. A relativistic kinetic equation for the case of Coulomb interaction is set up, corresponding to the non-relativistic equation of L. D. Landau [Ž. Eksper. Teoret. Fiz. 7 (1937), 203]. This is used to calculate the rate of transfer of energy and momentum between two gases. Application is made to the case of a monochromatic beam of ions entering an electron gas.

N. Rosen.

Longuet-Higgins, H. C.; and Pople, J. A. Transport properties of a dense fluid of hard spheres. J. Chem. Phys. 25 (1956), 884–889.

A simple approximate theory is developed for the evaluation of transport properties of a fluid composed of hard spheres. The principal assumptions are: (a) only binary collisions are considered, (b) the velocity distribution function for a molecule is Maxwellian about the local mean fluid velocity, with a spread depending only on the local temperature, and (c) the distribution function for the relative positions of pairs of molecules in space depends on the temperature and density of the fluid but not on the temperature gradient or the local shear. These assumptions permit the authors to give compact derivations of expressions for the shear viscosity, bulk viscosity, and thermal conductivity coefficients. A different procedure is used for the calculation of the self-diffusion coefficient. If $\mathbf{v}(t)$ is the velocity function of a molecule at time t, the auto-correlation function

$$\varphi(t_1, t_2)=\langle \mathbf{v}(t_1)\cdot \mathbf{v}(t_2)\rangle$$

is formed, the scalar product being averaged over the statistical distribution of the molecules. The assumption is then made that $\varphi(t_1, t_2)$ is an exponentially decaying function of $|t_1-t_2|$. The relationship to the Enskog theory based on the Boltzmann transport equation is discussed.

E. L. Hill (Minneapolis, Minn.).

Elasticity, Visco-elasticity, Plasticity

Goldberg, Martin A. Investigation of the temperature distribution and thermal stresses in a hypersonic wing structure. J. Aero. Sci. 23 (1956), 981–990.

The author presents here an engineering approach to calculating the transient temperature distribution and thermal stresses in an idealized multi-cell wing structure of a typical high speed aircraft. (The word "hypersonic" used in the title of this paper seems to have little bearing on the over-all analysis.) The heat flux into the system, resulting from aerodynamic heating, is assumed to vary sinusoidally with time and to remain constant with position. The analysis is then carried out by considering the heat conduction through the structure members. The effects of the geometric configuration of the members on the solution are discussed in detail.

T. Y. Wu.

See also: de Vito, p. 749; Kunze, p. 779.

Fluid Mechanics, Acoustics

Pritchard, J. Laurence. The dawn of aerodynamics. J. Roy. Aero. Soc. 61 (1957), 149–180.

A non-technical history of the theory and practice of flight from Newton to the Wright brothers, with many illustrations of apparatus.

Crease, J. Long waves on a rotating earth in the presence of a semi-infinite barrier. J. Fluid Mech. 1 (1956), 86–96.

The problem described in this interesting paper arises in connection with the propagation of tides and storm surges in the ocean. The author considered the long gravity waves approaching a semi-infinite barrier which extends parallel to the wave front, the whole system being under the influence of earth rotation. The problem is formulated by using the long-wave theory, the surface wave is assumed to have a simple harmonic time motion. By imposing an appropriate radiation condition on the downstream side, the resulting integral equation is solved by the Wiener-Hopf method. A part of the solution represents the usual diffraction phenomenon that, when the rotation is zero, there is a shadow region behind the barrier in which the disturbance diminishes rapidly with distance from the edge. However, the rotation effect gives rise to an additional wave in the shadow region; this wave has its front perpendicular to the incident wave, and travels along the barrier without attenuation in that direction. The amplitude of this wave may exceed that of the incident wave, but it falls off exponentially with distance from the barrier.

T. Y. Wu.

Campbell, I. J. The transverse potential flow past a body of revolution. Quart. J. Mech. Appl. Math. 9 (1956), 140–142.

In this compact and neat paper, the author considers the potential flow of an incompressible inviscid fluid past a body of revolution, not necessarily slender, set with its axis of rotation normal to the stream. At any point on the surface of the body the flow velocity is resolved into two components: w_1, directed along the meridian, and w_2, perpendicular to the meridian; the azimuthal angle θ is chosen to be zero in the direction of the approaching stream. By using an argument of the invariance property of the problem under rotation about the body axis, the author proves the following separation of variables

$$w_1(x, \theta)=w_1(x, 0)\cos\theta,\quad w_2(x, \theta)=w_2\left(x, \frac{\pi}{2}\right)\sin\theta,$$

where x is the body axis of rotation. Some quite trivial, but not so unimportant points were apparently left out in the proof.

T. Y. Wu (Pasadena, Calif.).

Davies, D. R.; and Bourne, D. E. On the calculation of heat and mass transfer in laminar and turbulent boundary layers. I. The laminar case. Quart. J. Mech. Appl. Math. 9 (1956), 457–467.

In a paper by the reviewer [Proc. Roy. Soc London Ser A 202 (1950), 359–377; MR 12, 218] laminar heat transfer in incompressible flow was studied by using the von Mises form of the temperature equation, with the relation between velocity and stream function approximated by the ($\frac{1}{2}$)th power law to which it asymptotes near the wall. The resulting expression for heat transfer was the exact asymptotic value as the Prandtl number $\sigma \to \infty$, since then the temperature boundary layer becomes so thin that the assumed relation is accurate throughout it. Even for $\sigma=0.7$, the value typical of most gases, the errors were only from 3 to 18 percent for typical laminar layers. The present paper uses a power law with smaller exponent than $\frac{1}{2}$ in the relation mentioned, to make it agree better in the outer part of the boundary layer. The results so obtained will be less accurate at high Prandtl numbers, but are more accurate, particularly for boundary layers in accelerating flow, when $\sigma=0.7$.

M. J. Lighthill (Manchester).

Davies, D. R.; and Bourne, D. E. On the calculation of heat and mass transfer in laminar and turbulent boundary layers. II. The turbulent case. Quart. J. Mech. Appl. Math. 9 (1956), 468–488.

Heat transfer across a turbulent boundary layer is calculated by (i) using a power-law fit to the profiles of the mean velocity and the eddy viscosity (assumed equal to the eddy diffusosity of heat) in the fully turbulent part of the layer (where molecular viscosity and conductivity are negligible), and then solving the von Mises form of the temperature equation in this region by means of the same mathematics as in the previous paper, and (ii) using the assumption of no variation of shear stress and heat flow across the inner "laminar" and "transition" sub-layer, together with the known mean-velocity distribution therein, to derive a solution which can be joined on to the previously mentioned solution. Satisfactory agreement is obtained with the measurements of Éliás [Z. Angew. Math. Mech. 10 (1930), 1], which were also explained satisfactorily by the theory of von Kármán [Proc. 4th Internat. Congress Appl. Mech., Cambridge, Eng., 1934, Cambridge, 1935, pp. 54–91, esp. pp. 77–83]. The method is applied also to the case of evaporation, and satisfactory agreement with experiments by Davies and Walters [Proc. Phys. Soc. Sect. B. 65 (1952), 640–645] is again found.

M. J. Lighthill (Manchester).

Glauert, M. B. A boundary layer theorem, with applications to rotating cylinders. J. Fluid Mech. 2 (1957), 89–99.

If, in a given solution of the boundary layer equation, the position of the wall is varied, then additional solutions of the boundary layer equations may be deduced. The theorem considers the nature of such solutions, for the general case of time-dependent three-dimensional compressible flow.

Applications of the theorem arise in several different fields, and it is shown that useful quantitative results can often be obtained with the minimum of calculation. In this paper, chief attention is focused on the case of a rotating circular cylinder, and explicit formulae are developed for the skin friction, valid for sufficiently low rotational speeds. The important results which the theorem gives for slip flow have been noted by previous authors, and only a brief discussion is given here, but certain extensions to these previous treatments are made. Other applications of the theorem are briefly mentioned. (From the author's summary.)

E. Leimanis.

Townsend, A. A. The properties of equilibrium boundary layers. J. Fluid. Mech. 1 (1956), 561–573.

Using only assumptions of similarity, the author shows that an equilibrium (i.e. self-preserving) turbulent boundary layer can exist only if the free stream velocity varies as a power of distance downstream with an exponent greater than (−1/3) and if the velocity defect from the free stream is small. Assuming further that the effective eddy viscosity is independent of distance from the wall over the outer part of the layer, he shows that most of the properties of equilibrium layers can be computed from the known behaviour of layers in zero pressure gradient. The predicted values of skin friction and the predicted shape and magnitude of the mean velocity distribution are in fair agreement with the observations of Clauser [Advances in applied mechanics, v. 4, Academic Press, New York, 1956, pp. 1–51]. (From author's summary.)

Many of the results of this paper have been described previously, in somewhat greater detail, in the author's book "The structure of turbulent shear flow" [Cambridge, 1956; MR 17, 1249].

D. W. Dunn (Ottawa, Ont.).

Nocilla, Silvio. Campi di moto transonici attorno a profili alari: applicazioni. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 90 (1955–56), 311–331.

Having drawn attention, in reviews of this author's previous papers [same Atti 89 (1954–55), 296–322; 90 (1955–56), 46–62; MR 17, 1023, 1250] to the importance of their contribution in applying and extending the method of Tomotika and Tamada [Quart. Appl. Math. 9 (1951), 129–147; MR 13, 179] for constructing transonic flows by a hodograph procedure, the reviewer contents himself with noting that several examples are treated in the present paper, and the numerical results compared with those given by more approximate procedures, the latter not, in general, coming well out of the comparison.

M. J. Lighthill (Manchester).

Imai, Isao. Application of the M^2-expansion method to compressible flow past isolated and lattice aerofoils. J. Phys. Soc. Japan 12 (1957), 58–67.

This paper represents a really marked simplification of the Janzen-Rayleigh method of evaluating subsonic flows around aerofoils. By use of an improved complex-variable treatment the author finally obtains simple formulas by which, given only the velocity distribution around the surface of an aerofoil for zero Mach number M, one can by quadratures alone obtain the terms in M^2 and M^4 in the expansion of the surface velocity distribution in powers of M.

The same is done, up to M^2 only, for a "lattice" (or "cascade") of aerofoils.

M. J. Lighthill.

Hall, W. B.; and Orme, E. M. Flow of a compressible fluid through a sudden enlargement in a pipe. Proc. Inst. Mech. Engrs. 169 (1955), 1007–1015, communications 1016–1020.

This paper develops a theory for the compressible flow through a sudden enlargement in a pipe. A one-dimensional type of analysis is used, and it is assumed that the

pressure across the face of the enlargement is equal to the pressure in the smaller pipe just before the enlargement. Experimental measurements, with air as the working fluid, agree well with theoretical predictions for the relation between throat Mach number and the Mach number downstream of the sudden enlargement for the range of enlargement ratios 0.0292–0.255 and for throat Mach numbers up to 1.00. (From authors' summary.)

D. W. Dunn (Ottawa, Ont.).

Woods, L. C. Generalized aerofoil theory. Proc. Roy. Soc. London. Ser. A. 238 (1957), 358–388.

The general problem of the steady flow of an inviscid gas, at subsonic speeds, past a porous aerofoil is solved. To do this the author makes certain assumptions: (1) the gas is replaced by a Kármán-Tsien tangent gas, (2) the mass flow through the porous aerofoil is a linear function of the pressure difference across the wall, (3) the mass of air sucked through the aerofoil is relatively small. However, the theory given is applicable to aerofoils of general shape placed at arbitrary incidence. Blasius's theorem is generalised to include the case of a compressible fluid. Several special cases of the above very general theory are given. These include (1) the flow about a given aerofoil and (2) the flow about bluff bodies to which finite bubbles adhere.

The author gives a new treatment of the steady, compressible, subsonic flow past a given aerofoil. Instead of determining, in the usual way, the incompressible flow past an initially unknown distorted profile the compressible problem is tackled directly using the given circulation and profile shape. Whichever method is used an integral equation must be solved at some stage of the calculation and the author claims that the direct solution presented is simpler than the usual technique.

G. N. Lance (Southampton).

Landahl, Mårten T. Unsteady flow around thin wings at high Mach numbers. J. Aero. Sci. 24 (1957), 33–38.

The reviewer's simple solution for unsteady flow about thin airfoils at high Mach number M [same J. 20 (1953), 402–406; MR 16, 419], which gives pressure coefficients including terms of order δM^{-1}, δ^2 and $\delta^3 M$, where δ is a parameter characterizing the scale of the disturbance, is taken as the first-order approximation in an "inverse Janzen-Rayleigh procedure", or expansion in powers of M^{-2}. From the higher approximations in the sequence, expressions for additional terms in the pressure coefficient, of order δM^{-3}, δM^{-5} and $\delta^2 M^{-2}$, are found. With these included, the results apparently become reliable down to $M=2$ in practical cases and lead to a satisfactorily simple procedure for flutter calculations.

M. J. Lighthill (Manchester).

Phillips, O. M. The intensity of Aeolian tones. J. Fluid Mech. 1 (1956), 607–624.

The general form of the theoretical aerodynamic sound field of a flow with solid boundaries [N. Curle, Proc. Roy. Soc. London. Ser. A. 231 (1955), 505–514; MR 17, 681] is specialized to the case of the Aeolian tones produced by the flow past a circular cylinder. The expected sound field is of dipole character, associated with the fluctuating force between the fluid and the cylinder. Many experiments have shown that the sound produced is of the same frequency as the fluctuations in lift, while there is negligible sound generated with double this frequency (corresponding to the fluctuations in drag). The present author analyses experiments by Kovásznay [ibid. 198 (1949), 174–190] at a Reynolds number of 50, for which a regular Karman vortex street occurs, and infers by a new technique, involving integration of vorticity in the wake, that the lift coefficient fluctuated with amplitude 0.76, which was ten times the fluctuation in drag coefficient, agreeing with the frequency observations noted above and with the observation by Gerrard [Proc. Phys. Soc. Sect. B. 68 (1955), 453–456] that the directional pattern of the sound field was close to that of a dipole with its axis normal to the direction of motion.

Experimental determinations of sound intensities at Reynolds numbers between 110 and 160 are reported. Evidence is presented that at these Reynolds numbers the fluctuations in lift are not in phase along the cylinder. The correlation distance is taken as 17 times the diameter (from some fairly crude observations), and the amplitude of lift fluctuations the same as at $R=50$. The predicted distribution of intensity then agrees well with the observations, varying in particular as U^6 when only the flow velocity U is altered.

For the turbulent flows occurring at higher Reynolds number, intensity measurements of Gerrard and of Holle [Akust. Z. 3 (1938), 321–331] are fitted to the same type of curve as in the laminar case, but with the factor of proportionality divided by 7, a reduction attributed partly to reduction in amplitude of the lift fluctuations, and partly to reduction in their correlation distance along the cylinder.

In fitting Gerrard's points to this curve the author has omitted one series of points, and also has rejected Gerrard's own conclusion that in his experiments intensity varied as the second (rather than the first) power of cylinder length, which would have implied that fluctuations are well correlated all along the wire. However, in this conclusion Gerrard was at variance with Holle, as well as with all the evidence on correlation of fluctuations, and the good agreement among different series of experiments and a firmly-based theory which the present author has achieved indicates that the bold discarding of a single series of Gerrard's points may have been the right decision.

M. J. Lighthill (Manchester).

See also: Langer, p. 738; Möglich, p. 781.

Optics, Electromagnetic Theory, Circuits

Heine, V. The thermodynamics of bodies in static electromagnetic fields. Proc. Cambridge Philos. Soc. 52 (1956), 546–552.

L'Auteur part de l'expression du travail des forces électromagnétiques

$$(1)\qquad dw = \int dv(\mathbf{E}\cdot d\mathbf{D}) + \int dv(\mathbf{H}\cdot d\mathbf{B})$$

et en déduit l'expression de l'énergie libre et de l'entropie. Il suppose que le volume des corps est constant et qu'il n'y a pas d'hystérésis.

Au lieu de conserver le champ total $(\mathbf{E}, \mathbf{B})$ il modifie les formules de manière à ne conserver que les champs appliqués d'origine extérieure $(\mathbf{E}_0, \mathbf{B}_0)$. Ce n'est pas seulement le champ influençant les systèmes donnés de charges et de courants, mais aussi le champ dû aux autres corps du système.

Il considère d'abord une substance polarisable et obtient l'expression

$$(2)\quad dw=\int_{\text{all space}} dv[(\mathbf{E}_0\cdot d\mathbf{D}_0)+(\mathbf{H}_0\cdot d\mathbf{B}_0)]$$
$$-\int_v dv(\mathbf{P}\cdot d\mathbf{E}_0)+\int_v dv(\mathbf{B}_0\cdot d\mathbf{M})$$

il calcule ensuite dF par $dF=dw$ quand $T=\text{const}$; puis S par $dF=dw-S\cdot dT$ et trouve

$$(3)\quad S=\int_v dv\int_{0\atop T=\text{const}}^{\mathbf{E}_0}\left(\frac{\partial\mathbf{P}}{\partial T}\cdot d\mathbf{E}_0\right)+\int_v dv\int_{T=\text{const}}\left(\frac{\partial\mathbf{M}}{\partial T}\cdot d\mathbf{B}_0\right).$$

Il examine ensuite le cas d'une substance polarisable dans le champ d'un aimant permanent et obtient l'expression

$$(4)\quad -\tfrac{1}{2}\int_{v'} dv\int_0^{\mathbf{B}_0}(\mathbf{M}_0\cdot d\mathbf{B}_0)-\int_v dv\int_{0\atop T=\text{const}}^{\mathbf{B}_0}\mathbf{M}\cdot d\mathbf{B}_0.$$

Malgré que cette expression (4) soit différente de (2) elle conduit encore à l'expression (3) de l'entropie. Il en conclut qu'une formule donnée par Guggenheim doit être inexacte.

Toutes ces questions sont présentées d'une manière assez différente dans l'ouvrage de l'auteur de cette analyse [Electrostatique et magnétostatique, Masson, Paris, 1953, pp. 604, 617, 705, 732; MR 16, 99].

E. Durand (Toulouse).

Bodiou, Georges. Une forme spinorielle des équations de l'électromagnétisme. C. R. Acad. Sci. Paris 243 (1956), 1287–1289.

Il s'agit d'une transcription des équations de Maxwell-Lorentz en termes de spineurs du premier rang, dans l'espace temps. Tout bivecteur B réel est le dual de la somme des "densités de moment électromagnétique" de deux spineurs.

$$(1)\qquad B=B(\psi_1,\psi_1)+B(\psi_2,\psi_2)$$

avec

$$(2)\qquad \psi_1=\varphi_1+\varphi_1^-\qquad \psi_2=\varphi_2+\varphi_2^-.$$

Les équations de Maxwell-Lorentz $\partial_l B^{ll}=J^l$ et $\partial_l DB^{ll}=0$ équivalent à

$$(3)\qquad \partial_l(B-DB)^{ll}=J^l$$

(jointe à l'équation conjuguée).

La solution générale est décrite sous la forme (1) et (2) à partir de deux demi-spineurs φ_1 et φ_2 solutions de

$$(4)\qquad S(\varphi_1, d\varphi_1)+S(\varphi_2, d\varphi_2)=0,$$

$$(5)\qquad \begin{cases}8S(\varphi_1,\varphi_2)\times\nabla\times\varphi_1=J\times\varphi_2,\\ 8S(\varphi_1,\varphi_2)\times\nabla\times\varphi_2=J\times\varphi_1.\end{cases}$$

Dans le vide, toute onde électromagnétique se décrit ainsi d'une infinité de façons, comme duale de la somme des densités des moments électromagnétiques de deux corpuscules de Dirac à masse nulle; les densités de spin sont nulles. *E. Durand* (Toulouse).

Kunze, Günther. Zur Röntgenstreuung an unvollständigen zylindrischen Gittern. I, II. Acta Cryst. 9 (1956), 841–847, 847–854.

I. The amplitude of X-rays scattered by a number of atoms equally spaced along an arc of a circle is expressed in the form of a series of Bessel functions

$$R(r^*,\phi^*)=p^{-1}NF_\nu\sum_n\sum_k\frac{\sin n\pi/p}{n\pi/p}\exp[in(\phi^*-\phi_0-\pi/p)]$$
$$\times\exp[iNk(\phi^*-\gamma)]\times(-i)^{Nk+n}J_{Nk+n}(2\pi r^*r_\nu),$$

where r, ϕ are polar coordinates; r^*, ϕ^* the corresponding reciprocal coordinates; the arc extends from ϕ_0 to $\phi_0+2\pi/p$; N is the number of atoms in a complete circle; r_ν is the radius of the circle; F_ν is the atomic scattering factor; and γ appears to be ϕ_0+ the angle equivalent to half the interatomic distance on the arc. For a complete circle ($p=1$) the double sum degenerates into a single sum over k, since only the terms with $n=0$ are non-zero. The expression is generalized to a structure with several types of atoms with differing F_ν and R_ν, and with arcs starting at slightly different values of ϕ_0. The theory is applied in particular to the form of the 060 reflexion from chrysotile.

II. The angle at which the maximum intensity of reflexion from a cylindrical lattice occurs is displaced somewhat from the Bragg angle for the corresponding plane lattice, the displacement being a complicated function of the parameters introduced in I. For a complete cylinder and $0k0$ reflexions Bragg's law reduces to

$$\frac{2\sin\theta}{\lambda}=\frac{k+\Delta}{b},$$

where k is the order of the reflexion, b is the spacing along the arc, and $\Delta\sim 0.8086k^{\frac{1}{3}}[b/2\pi(r+t)]^{\frac{2}{3}}$, where r is the inner radius and t is the thickness of the bent lattice.

Models of actual structures (e.g. for antigorite, kaolin) with bent lattices and superstructures are discussed, and expressions are discussed for the amplitude of low-angle scattering.

In both I and II much use is made of convolutions (Faltungsoperationen). *A. J. C. Wilson* (Cardiff).

★ **Barnett, R. I., Jr.; and Tai, C. T. A study of an open rectangular waveguide partly filled with a stratified dielectric.** Contract No. AF 19(604)–1725. AFCRC-TN-56-599-ASTIA Document No.: AD 98805. The Ohio State University Research Foundation, Columbus, Ohio, 1956. iii+9 pp.

Vacca, Maria Teresa. Onde magneto-idrodinamiche in un fluido elettricamente conduttore entro un tubo indefinito a sezione rettangolare. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 90 (1955–56), 633–646.

In this paper is discussed the propagation of magneto-hydrodynamical waves in a homogeneous isotropic incompressible viscous electrically-conducting fluid contained in an infinite straight pipe with rectangular cross-section. The fluid is acted on by an external uniform magnetic field $\mathbf{H}_0$ directed along the axis of the pipe. The solution sought is of the form $\mathbf{H}=\mathbf{H}_0+\mathbf{h}$, the velocity of a typical particle of the fluid being proportional to $\mathbf{h}$. A similar problem for a pipe with a circular cross section was discussed by C. Agostinelli [Univ. e Politec. Torino. Rend. Sem. Mat. 15 (1955–56), 107–120; MR 18, 699].

E. T. Copson (St. Andrews).

Cowling, T. G. The dissipation of magnetic energy in an ionized gas. Monthly Not. Roy. Astr. Soc. 116 (1956), 114–124.

One generally supposes that the current density $\mathbf{j}$ in an ionized gas moving in an external magnetic field $\mathbf{H}$ can be expressed in the form

$$(*)\qquad \mathbf{j}=\sigma_0\mathbf{E}'_\parallel+\sigma_1\mathbf{E}'_\perp+\sigma_2\mathbf{H}\times\mathbf{E}'_\perp/H,$$

where $\mathbf{E}'_\parallel$ and $\mathbf{E}'_\perp$ are respectively the components parallel and perpendicular to the magnetic field of the electric force on the moving material and σ_0, σ_1 and σ_2 are appropriate conductivities. The electric field $\mathbf{E}'$ is

related to the field $\mathbf{E}$ on matter at rest and the velocity $\mathbf{v}$ by $\mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{H}$. Using these relations, Cowling shows that the dissipation of the magnetic energy per unit volume and time is given by $\mathbf{E}'\cdot\mathbf{j}$: this does not include the decrease in the magnetic energy due to the action of the Lorentz force $\mathbf{j}\times\mathbf{H}$. From (*) it now follows that the dissipation is given by j^2/σ_0 if $\mathbf{j}$ is parallel to $\mathbf{H}$ while it is given by j^2/σ_3, where $\sigma_3=\sigma_1+\sigma_2^2/\sigma_1$ if $\mathbf{j}$ is perpendicular to $\mathbf{H}$.

When a pressure gradient is present then the effect of the diffusion of electrons relative to the ions is equivalent to the presence of an electric force $\mathbf{E}''=(\operatorname{grad} p_e)/ne$, where p_e denotes the electron pressure and n is the number density of electrons. Under these circumstances, there is the additional term $-\mathbf{E}''\cdot\mathbf{j}$ in the expression for the dissipation. Cowling points out there is no universally valid reason for this term being of one sign rather than another. However, physical arguments are advanced which suggest that this term may be expected to vanish on the average.

The electric currents in a partially ionized gas are treated ab initio and the author obtains the expression

$$(**)\quad ne\mathbf{E}'=\frac{\kappa\kappa_e+\kappa\kappa_i+\kappa_i\kappa_e}{\kappa_i+\kappa_e}H\mathbf{j}+(1-2\beta F)\mathbf{j}\times\mathbf{H} -\frac{1}{2}\{1-f-2\beta(F-f)\}\operatorname{grad} p -\frac{F}{H(\kappa_i+\kappa_e)}\{F(\mathbf{j}\times\mathbf{H})\times\mathbf{H}-(F-f)\operatorname{grad} p\times\mathbf{H}\}$$

for the current density. In this equation f and F denote the fractional contributions of the neutral atoms to the total pressure and total density, $\kappa=(\omega_e\tau)^{-1}$, $\kappa_e=(\omega_e\tau_e)^{-1}$, $\kappa_i=(\omega_i\tau_i)^{-1}$ and $\beta=\kappa_e/(\kappa_i+\kappa_e)$ where ω_e and ω_i are the Larmor frequencies (of spiralling of the electrons and ions in the magnetic field), τ and τ_e are the mean times between successive collisions of an electron with ions and neutral atoms, respectively, and τ_i is $m_{\text{atom}}/(m_{\text{atom}}+m_{\text{ion}})$ times the interval between successive collisions of an ion with neutral atoms. In terms of this equation the coefficients σ_0 and σ_3 which occur in the earlier discussion are shown to have the values

$$\sigma_0=ne(\kappa_i+\kappa_e)[H(\kappa\kappa_i+\kappa\kappa_e+\kappa_i\kappa_e)]^{-1},$$
$$\sigma_3=\sigma_0[1+F^2(\kappa\kappa_i+\kappa\kappa_e+\kappa_i\kappa_e)]^{-1}]^{-1}$$

if the contribution from the term in $\operatorname{grad} p$ in (**) is neglected as vanishing on the average. Applications of these results to the dissipation of magnetic energy in interstellar space is discussed. *S. Chandrasekhar.*

Larenz, R. W. Zur Magneto-Hydrodynamik kompressibler Medien. Z. Naturf. **10a** (1955), 761–765.

This paper discusses the equations of motion appropriate for an ionized gas when a magnetic field prevails. The equations are derived in the usual manner in which the effects of electrical conductivity appear as a consequence of the collisions between atoms, ions and electrons which on the average destroy the relative momenta of the colliding particles. The equations derived by the author are not new [cf. A. Schlüter, Z. Naturf. **5a** (1950), 72–78; **6a** (1951), 73–78; see also the paper reviewed above]. However, the author shows that for adiabatic flow the equations can be reduced to a set of three equations: an equation of motion involving a current density and an equation (in addition to Maxwell's equation) for the current density involving the vector potential and pressure gradients. *S. Chandrasekhar.*

Lovass-Nagy, V. On an application of Egerváry's hypermatrix-algorism to the mathematical investigation of polyphase transformers. Acta Tech. Acad. Sci. Hungar. **15** (1956), 261–286. (Russian, French and German summaries)

The purpose of the paper is to solve systematically the linear differential equations describing the steady-state and transient behavior of single- and poly-phase transformers. As a main tool the author employs theorems on the spectral decomposition of special partitioned matrices. [The main reference here is E. Egerváry, Acta Sci. Math. Szeged **15** (1954), 211–222; MR **16**, 327].

The mathematical problem is to solve (*) $\mathbf{L}d\mathbf{i}/dt+\mathbf{R}\mathbf{i}=\mathbf{u}$, where $\mathbf{L}$, $\mathbf{R}$ are constant square matrices of order n, with $\mathbf{R}$ diagonal, and where $\mathbf{i}$, $\mathbf{u}$ are n-vectors depending on t. The author solves (*) with matrix analogs of the linear integration formula for $n=1$. *G. E. Forsythe.*

See also: Greniewski, p. 712; Moisil, p. 712; Malkevič, p. 767; Möglich, p. 781; Manarini, p. 781; Arcidiacono, p. 782; Moisil, p. 784.

Thermodynamics and Heat

Gerlach, Johannes. Über absolute Temperatur und Entropie bei einfachen Systemen. Ann. Univ. Sarav. **5** (1956), 112–127 (1957).

In this paper thermodynamics is treated as a formal theory, whose logical structure is exhibited in close analogy with Carathéodory's treatment.
N. G. van Kampen (Utrecht).

Ritchie, R. H.; and Sakakura, A. Y. Asymptotic expansions of solutions of the heat conduction equation in internally bounded cylindrical geometry. J. Appl. Phys. **27** (1956), 1453–1459.

In certain heat conduction problems it is necessary to find the solution of the equation

$$(*)\qquad K\nabla^2 v=\partial v/\partial t\quad (K=\text{const})$$

in cylindrical coordinates (r, θ, z) in a region $r\geqq a$ with given boundary conditions on the inner boundary $r=a$. If the problem is two-dimensional and has cylindrical symmetry, it may be solved by the Laplace transform technique; one puts

$$\bar{v}=\int_0^\infty e^{-pt}v(r, t)dt.$$

Then

$$v=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} e^{pt}\bar{v}dt,$$

where

$$\bar{v}=AK_0[r\sqrt{(p/K)}]+BI_0[r\sqrt{(p/K)}].$$

If $v\to 0$ as $r\to\infty$, B is zero, and A is determined by the conditions on $r=a$. The problem discussed here is the behaviour of the resulting solution for large values of t.

The problem of the flow of a gas through a porous medium leads to a non-linear equation of the form (*) where K is a function of v. In most cases, the value of K is nearly constant, so that the methods developed here give a first approximation to this non-linear problem.
E. T. Copson (St. Andrews).

See also: Rytov, p. 775; Goldberg, p. 776; Heine, p. 778.

Quantum Mechanics

Angelitch, T. P. On the determination of the angular momentum operator in quantum mechanics. Bull. Soc. Math. Phys. Macédoine 6 (1955), 30–34. (Serbo-Croatian. English summary)

An expression is derived for the square of the angular momentum operator which expression is claimed to hold in any arbitrary coordinate system. The formula is

$$M^2 = -\hbar^2 \delta_{pq}{}^{jk} g_{mj} r^m \frac{D}{Dx^k}\left(r^p g^{ql} \frac{D}{Dx^l}\right)$$

where r^i are the contravariant coordinates of the position vector, g_{ij} is the fundamental metric tensor of the Euclidean three dimensional space, D/Dx^i the covariant derivatives with respect to the general coordinate x^i, and $\delta_{pq}{}^{jk}$ the Kronecker symbol of the fourth order.

M. J. Moravcsik (Upton, N.Y.).

★**Février, Paulette. Déterminisme et indéterminisme.** Presses Universitaires de France, Paris, 1955. xii+250 pp. 1000 francs.

Wightman, A. S. Quantum field theory in terms of vacuum expectation values. Phys. Rev. (2) 101 (1956), 860–866.

This paper attempts to deduce some general properties of vacuum expectation values of field operators from very general principles, such as Lorentz invariance, absence of negative energy states and local commutativity of field operators. An important consequence is that these expectation values are boundary values of analytic functions. The local commutativity property leads to a spectral representation for the two-fold expectation value. Finally, it is shown how to construct a theory of a scalar field $\phi(x)$, given the analytic functions $F^{(n)}$ whose boundary values are taken to be the n-fold vacuum expectation values of $\phi(x)$, provided the $F^{(n)}$ satisfy the conditions discussed previously. {The developments of the ideas outlined in this paper will probably provide a very important tool for a reformulation of field theory in a rigorous form. Relations of the type of dispersion relations will probably also come out from the developments of this work.}

M. Cini (Torino).

Malenka, B. J.; and Primakoff, H. Isotopic spin and antinucleon-nucleon scattering. Phys. Rev. (2) 105 (1957), 338–343.

The first part of the paper is dedicated to the construction of the total isotopic spin operator for a system of nucleons and antinucleons by means of quantized field theory. It is found that the usual assumption, according to which the eigenvalues of T_3 for antiproton and antineutron are $-\frac{1}{2}$ and $\frac{1}{2}$ respectively if the eigenvalues associated with proton and neutron are assumed $\frac{1}{2}$ and $-\frac{1}{2}$, is correct. The total T_3 is therefore obtained by summing the third component eigenvalues of isospin of particles and antiparticles. This result is applied to obtain the eigenfunctions in configuration space of the two body system (nucleons and antinucleons).

An application to nucleon-antinucleon scattering leads to some predictions for the ratio of elastic charge exchange to elastic noncharge exchange scattering of an antiproton by a proton, and for the ratio of elastic noncharge exchange of an antiproton by a neutron and by a proton. The simplifying assumptions are, however, very questionable, because they amount to taking lowest order perturbation results.

M. Cini (Torino).

Möglich, F. Zur Hydrodynamik wirbelfreier Elektronenfelder. Ann. Physik (6) 18 (1956), 230–236.

The hydrodynamical equations of motion are considered for a fluid of charged particles. The interaction is inserted as the vector and scalar potentials produced by the fluid itself but its physical effects are not discussed. In the case where the velocity field is irrotational the equations of motion are brought into canonical form. A change of canonical variables is further carried out leading to an equation which by neglecting of one term reduces to the Schrödinger equation for a charged particle in an electromagnetic field. The relation of this result to the work of Bohm [Phys. Rev. (2) 85 (1952), 166–179, 180–193; MR 13, 709] and others on a causal reinterpretation of quantum mechanics is indicated.

L. Van Hove (Utrecht).

Ivanović, Dragiša M. Theory of motion of neutrons through the mixture of elements. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. 2 (1956), 43 pp. (Serbo-Croatian. English summary)

See also: Toll, p. 735; Wigner, p. 771.

Relativity

Manarini, Anna Marisa. Un teorema di unicità per le equazioni di Maxwell-Minkowski. Boll. Un. Mat. Ital. (3) 11 (1956), 440–444.

The Maxwell-Minkowski electromagnetic equations for a homogeneous isotropic conductor, with no rest charge and moving with velocity v relative to an inertial system, are the usual Maxwell equations together with the relations,

$$\mathbf{D}+\mathbf{v}\times\mathbf{H}/c=\varepsilon(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c),$$
$$\mathbf{B}-\mathbf{v}\times\mathbf{E}/c=\mu(\mathbf{H}-\mathbf{v}\times\mathbf{D}/c),$$
$$\mathbf{J}=\sigma(1-v^2/c^2)^{-\frac{1}{2}}(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c),$$

where ε, μ, σ are respectively the dielectric constant, the permeability, and the conductivity of the conductor. It is here proved that these equations have a unique solution in a closed region of space, provided that the initial values of **E** and **H** are given at all interior points and the tangential components of **E** and **H** over the surface of the region are given.

A. J. McConnell (Dublin).

★**Romanas, Georges. Entraînement de l'éther et de ses ondulations par la matière en mouvement. Examen critique de la théorie de la relativité.** M. A. Petris, Athènes, Grèce, 1955. 31 pp.

Törnebohm, Håkan. Epistemological reflexions over the special theory of relativity and Milne's conception of two times. Philos. Sci. 24 (1957), 57–69.

Zel'manov, A. L. Chronometric invariants and co-moving coordinates in the general relativity theory. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 815–818. (Russian)

In cosmological theory, use is frequently made of

"co-moving coordinate systems" with respect to which the matter throughout the universe is at rest. In connection with transformations from one comoving coordinate system to another, one can define quantities which are invariant with respect to the time transformation (chronometric invariants) and covariant with respect to the transformations of the space coordinates. Various physical laws are expressed by means of such quantities. *N. Rosen* (Haifa).

Rozo M., Dario. New concept of Einsteinian relativity. Rev. Acad. Colombiana Ci. Exact. Fis. Nat. **9** (1956), 253–261. (Spanish)

v. Laue, Max. Von Kopernikus bis Einstein. Naturwiss. Rundschau **10** (1957), 83–89.

The purpose of this non-technical essay is "die allgemeine Relativitätstheorie, welche im Bewusstsein der Zeitgenossen meist als ein Randgebiet der Physik erscheint, in ihrer zentralen Stellung zu einem der Grundprobleme aller Naturwissenschaft herauszustellen".

Bastin, E. W.; and Kilmister, C. W. Eddington's statistical theory. I. Introduction. Rend. Circ. Mat. Palermo (2) **5** (1956), 187–203.

The authors give a detailed paragraph by paragraph exegetic account of the first five sections of Eddington's book, "Fundamental theory" [Cambridge, 1946; MR **11**, 144] and conclude that the statistical arguments Eddington used cause the greatest difficulties in these sections. *A. H. Taub* (Urbana, Ill.).

Singer, S. F. Application of an artificial satellite to the measurement of the general relativistic "red shift". Phys. Rev. (2) **104** (1956), 11–14.

A satellite describes a circular orbit around the earth. The rotation of the earth is neglected. The field of the sun is neglected and that of the earth is of the usual Schwarzschild form. The satellite carries an atomic clock, measuring proper time, and sends signals to an observer on the earth when it is passing over him. The problem is essentially to find the ratio s_0/s_1, where s_0 is the interval of proper time for the satellite between the emissions of signals and s_1 the interval of proper time for the observer between their receptions. Since ds/dt is constant for the satellite and for the observer, s_0/s_1 is simply the ratio of the two constant values of ds/dt, calculated first for the satellite and secondly for the observer. This ratio is easily found and yields the so-called shift

$$\Delta=1-\frac{s_0}{s_1}=m\left[\frac{3}{2}\frac{1}{R_E+h}-\frac{1}{R_E}\right],$$

where m is the mass of the earth, R_E its radius and h the height of the satellite above the earth's surface. The shift is red for $h<\frac{1}{2}R_E$ and violet for $h>\frac{1}{2}R_E$. {In order to bring out the fact that he is dealing essentially with an integrated effect, the above account differs slightly from that given by the author.} *J. L. Synge* (Dublin).

Arcidiacono, Giuseppe. Sul campo elettromagnetico generalizzato. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 631–636.

The author has shown [same Rend. (8) **18** (1955), 386–391, 515–519; MR **17**, 1244] that the generalised Maxwell equations in the theory of "final" relativity describe a field of ten components, six of which constitute the ordinary electromagnetic field, whilst the remaining four components describe a new field. The present paper completes the study of this generalised field by constructing the electromagnetic energy tensor of the field, determining the differential equations satisfied by the potentials, and finally showing that the equations can be obtained from a variational principle. *A. J. McConnell* (Dublin).

See also: Toll, p. 735; Rindler, p. 782.

Astronomy

Zoltán, Gábos. Contribution à l'étude du champ gravitationnel. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 191–199. (Romanian. Russian and French summaries)

The author applies one of the consequences of our theory of gravitation to characterize the gravitational field, by a quadrivector, where its time component is a scalar potential φ and the space component a vector potential $\beta\varphi\dot{r}/c^2$.

The motion is determined, according to the author, by the principle $\delta\int L\,dt=0$, where the Lagrange function has the form

$$L=m_0[-\beta/c^2+\beta\dot{r}^2/c^2-\phi].$$

If terms are retained only up to the second order in $1/r$, the resulting differential equation gives, in the case of the case of the planet Mercury, a trajectory, the perihelion of which has a displacement of 39″, 41 per century.

In the second part of his work, the author applies the same mechanical considerations to the movement of the photon, assuming that its energy and mass are respectively $h\nu$ and $h\nu/c^2$ and obtains well-known results for the red-shift of the spectrum as well as for the deviation of light in the gravitational field of the sun. *O. Onicescu.*

Kovalevsky, Jean. Sur la détermination des orbites par la méthode de Laplace (méthode des variations). C. R. Acad. Sci. Paris **244** (1957), 856–859.

A. Danjon [Bull. Astr. (2) **16** (1951), 85–110; MR **13**, 391] demonstrated the effectiveness of the Laplacian method of orbit determination by combining a method of iteration for obtaining a provisional orbit with a method of differential correction. The corrections to be evaluated are those to the basic data α_0, δ_0, α_0', δ_0', α_0'', δ_0'', the right ascension and declination and their first and second derivatives for a single date, t_0. This paper deals with the indeterminateness of the solution which arises with this choice of unknowns if the geocentric velocity lies in the plane Sun-planet-Earth. It is shown that the indeterminateness vanishes if the differential corrections are sought to α_0, δ_0, D_0, α_0', δ_0', D_0', in which D_0 is the geocentric distance of the planet at t_0. The author remarks that this choice of unknowns has advantages for general application. *D. Brouwer.*

Rindler, W. On the coordination of the Riemannian and kinematic techniques in theoretical cosmology, with particular reference to the shift-distance law. Monthly Not. Roy. Astr. Soc. **116** (1956), 335–350 (1957).

The paper is mainly concerned with the expression for the red-shift, in homogeneous and isotropic models of the universe, in terms of distance parameters and with the interpretation of the Humason, Mayall and Sandage data on the red-shifts of clusters of galaxies. Both general

relativity and kinematical relativity are dealt with and the author makes a distinction between a "theory" and a "technique." As far as the present reviewer can understand the distinction, a "technique" is the process of working out some specific prediction or consequence of a theory. The distance parameter, l, which is regarded as fundamental, is the distance of an object from the observer calculated instantaneously at the moment at which the light by which the object is eventually observed was leaving it. This distance is called "objective" and it is asserted that it would be measured by a chain of "fundamental" observers strung out between object and observer. This is not an operational definition since the chain of observers cannot in practice be set up by human scientists even from the earth to the moon, let alone to a galaxy. The exposition suggests by implication that time- and distance-scales are interrelated. This is an axiom of kinematical relativity but is not true in general relativity. When the interpretation of observations is reached, the author has to express the red-shift, z, in terms of the observationally significant luminosity-distance, L, which is not identical with l. The formula for z in terms of L is an infinite series and an interesting method of testing the validity of approximations to the series is given. Numerical evaluation of the first two coefficients in the series from the Humason, Mayall and Sandage data leads to 142 km/sec/mpc for the Hubble "constant" and to the conclusion that the expansion of the galaxies is being retarded. Sandage has obtained a value of about 180km/sec/mpc and the discrepancy is attributed — rightly in the reviewer's opinion — to the different ways of applying the least squares method.

G. C. McVittie.

See also: Cimino, p. 739; Cowling, p. 779; Singer, p. 782.

Geophysics

Jobert, Nelly. Sur la période propre des oscillations sphéroïdales de la Terre. C. R. Acad. Sci. Paris **244** (1957), 921–922.

Doodson, A. T. Tides and storm surges in a long uniform gulf. Proc. Roy. Soc. London. Ser. A. **237** (1956), 325–343.

Ignoring the tide generating forces within the gulf, the equations of motion of the tides within a long uniform gulf are integrated numerically to determine the semi-diurnal tide, and the effect of a storm surge on it, within the gulf.

The integration is carried out from the barrier to the mouth of the gulf, using an assumed shape at the barrier. Starting with a simple semi-diurnal tide gives rise to other constituents at the mouth because of the non-linear terms in the equations; adjustments are therefore made to the form at the barrier and the integration repeated until a tide principally semi-diurnal at the mouth is obtained.

The storm surge, represented by an expression of the form $S\exp(-t^2/16)$, is superimposed on the semi-diurnal tide, and the calculations carried out in the same way. Results are obtained for storm surges whose maxima occur at high water, low water, and the two times midway between these.

D. C. Gilles (Manchester).

Chorley, Richard J.; Malm, Donald E. G.; and Pogorzelski, Henry A. A new standard for estimating drainage basin shape. Amer. J. Sci. **255** (1957), 138–141.

Lemniscates are used to approximate the various shape of drainage basins. These curves are defined by their long diameter l and a factor K indicating the fullness of the shape. The diameter l is proposed to be identified with the length of the main stream of the area while K is determined such that the total length of the periphery is equal in the drainage basin and in the corresponding lemniscate. The effect of local irregularities in the shape of the stream axis and of the basin periphery is questionable.

H. A. Einstein (Berkeley, Cal.).

Jerie, H. G. A contribution to the problem of analytical aerial triangulation. Photogrammetric Engrg. **22** (1956), 40–49.

See also: Malkevič, p. 767; Crease, p. 776.

OTHER APPLICATIONS

Games, Economics

Bagemihl, Frederick. Transfinitely endless chess. Z. Math. Logik Grundlagen Math. **2** (1956), 215–217.

The possibility of an endless game of chess has been established [M. Euwe, Akad. Wetensch. Amsterdam Proc. 32 (1929), 633–642; M. Morse and G. A. Hedlund, Duke Math. J. **11** (1944), 1–7; MR **5**, 202]. The present author shows how such an endless game, given by a sequence of moves, may be continued transfinitely in such a way as not to run afoul of the German rules whereby a game is a draw if the same sequence of moves occurs three times in succession. The proof proceeds by a transfinite induction.

E. R. Lorch (New York, N.Y.).

Hoskin, Michael. The theory of games. Eureka no. **19** (1957), 5–10.

An expository article.

See also: Bellman, p. 744; Finney and Cope, p. 774.

Information and Communication Theory

Lindley, D. V. On a measure of the information provided by an experiment. Ann. Math. Statist. **27** (1956), 986–1005.

A measure is introduced of the information provided by an experiment. The measure is derived from the work of Shannon and involves the knowledge prior to performing the experiment, expressed through a prior probability distribution over the parameter space. The measure is used to compare some pairs of experiments without reference to prior distributions; this method of comparison is contrasted with the methods discussed by Blackwell. Finally, the measure is applied to provide a solution to some problems of experimental design, where the object of experimentation is not to reach decisions but rather to gain knowledge about the world. (From the author's summary.)

H. Teicher (Lafayette, Ind.).

Control Systems

Moisil, Gr. C. L'application des imaginaires de Galois à la théorie des mécanismes automatiques. III. Schémas à relais polarisés. Com. Acad. R. P. Romîne 5 (1955), 959–963. (Romanian. Russian and French summaries)

Moisil, Gr. C. Contribution à l'étude algébrique des mécanismes automatiques. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 183–230. (Romanian. Russian and French summaries)

Moisil, Gr. C. Les relations entre la méthode de Luntz et celle de Tzetline pour les schémas en pont. Com. Acad. R. P. Romîne 6 (1956), 743–744. (Romanian. Russian and French summaries)

There is indicated a connection between the matrix of A. G. Lunc [Dokl. Akad. Nauk SSSR (N.S.) **70** (1950), 421–423; **75** (1950), 201–204; MR **11**, 574; **12**, 779] and the matrix of M. L. Cetlin [ibid. **86** (1952), 525–528; MR **14**, 606] for the description of the scheme of a multipole.
B. Germansky (Jerusalem).

Moisil, Gr. C. Observations sur la note, "Methode des schémas équivalents pour l'étude des relais temporisés". Com. Acad. R. P. Romîne 5 (1955), 933. (Romanian)

Ioanin, Gh. Méthode des schémas équivalents pour l'étude des relais temporisés. Com. Acad. R. P. Romîne 5 (1955), 923–931. (Romanian. Russian and French summaries)

Ioanin, Gh. Sur la théorie algébrique des contacts multipositionnels et son application à l'étude des contacts réels. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 231–240. (Romanian. Russian and French summaries)

See also: Doetsch, p. 735.

HISTORY, BIOGRAPHY

★ **Wolfer, Ernst Paul. Eratosthenes von Kyrene als Mathematiker und Philosoph.** P. Noordhoff N.V., Groningen-Djakarta, 1954, iv+68 pp.

The first chapter of this inaugural dissertation gives a cogent argument that the lost Platonicus of Eratosthenes was actually a dialogue with Plato as the chief speaker. The second suggests that Eratosthenes is the originator of several important ideas about means, e.g. the classical definition of the harmonic mean. The third discusses the 'mesolabe', i.e. an instrument (especially important in the history of the duplication of the cube) which was invented by Eratosthenes for inserting any desired number of mean proportionals. The fourth gives a philological commentary on several terms occurring in the citations of Eratosthenes by Theon of Smyrna. The fifth describes traces of Eratosthenes in the works of Nicomachus, Iamblichus and Proclus. The sixth deals with the tradition of the mathematical works of Eratosthenes, while the seventh and final chapter discusses him as a philosopher, in particular as a systematizer of mathematics.
S. H. Gould (Providence, R.I.).

Shimomura, Torataro. Die Bildung der Mathematik in der Polis. Eine Idee zur metaphysischen Genealogie der Mathematik als Mathesis Universalis. Ann. Japan Assoc. Philos. Sci. 1 (1956), 1–31.

Western, A. E. Note on Fermat's methods of factorisation. Math. Gaz. 41 (1957), 56–57.

Subbotin, M. F. The works of H. Poincaré in celestial mechanics. Voprosy Ist. Estest. i Tehn. no. 2 (1956), 114–123. (Russian)

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